# The solution of pressure velocity coupling

Gergely Kristóf

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## The Ψ-ω method for steady flow

The Poisson equation for Ψ :

$$\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=-\omega$$

The vorticity transport equation:

$$\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x}-\frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y}=\nu\left(\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2}\right)$$

Segregated iteration:

$$\psi^0,\omega^0 \xrightarrow{Poisson} \psi^1,\omega^0 \xrightarrow{VTE} \psi^1,\omega^1 \xrightarrow{Poisson} \psi^2,\omega^1 \ \ldots$$

## Governing equations for incompressible flows

An iterative method will be used for solving the discrete forms of the governing equations.
Further potential simplification:
Segregated iteration. (Separate equation is solved for every field variable, in which every other field variable is treated as a constant value.)

Continuity:

$$\nabla\cdot\vec{v}=0$$

Navier-Stokes equation:

$$\frac{\partial u}{\partial t}+\nabla\cdot(u\vec{v})=-\frac{\partial p/\rho_0}{\partial x}+\nabla\cdot(\nu\nabla u)+g_x$$

$$\frac{\partial v}{\partial t}+\nabla\cdot(v\vec{v})=-\frac{\partial p/\rho_0}{\partial y}+\nabla\cdot(\nu\nabla v)+g_y$$

$$\frac{\partial w}{\partial t}+\nabla\cdot(w\vec{v})=-\frac{\partial p/\rho_0}{\partial z}+\nabla\cdot(\nu\nabla w)+g_z$$

This system is not suitable for segragated iteration.

## Pressure based solution
### The pressure equation

Continuity:

$$\nabla\cdot\underline{u}=0$$

The equation of motion in 2D, when g=0:

$$\frac{\partial \underline{u}}{\partial t}+\nabla\cdot(\underline{u}\otimes\underline{u})=-\nabla(p/\rho_0)+\nu\Delta\underline{u}$$

Shorthand notations: $P=p/\rho_0$ and $\underline{f}$.

$$\frac{\partial \underline{u}}{\partial t}=\underline{f}-\nabla P$$

Let's take the divergence of the equation of motion assuming: $\frac{\partial}{\partial t}\nabla\cdot\underline{u}=0$

$$\Delta P=\nabla\cdot\underline{f}$$

This is a Poisson equation for the pressure.

This is useful for calculating the pressure field also in Ψ-ω methods.

## Two possible solutions

- Ψ-ω method
  Eliminates the pressure from the equation of motion by the introduction of a potential punction.
- Pressure correction method
  A new equation for the pressure field is solved instead of the continuity eq.

## Can the Poisson equation for the pressure be a substitute for the continuity equation?

We apply the numerical Laplacian for discretizing the Poisson equation for the pressure, and we solve this:

$$\tilde{\Delta}P^n=\tilde{\nabla}\cdot f_i^n \qquad \text{(Eq.1)}$$

Than we apply the Euter method for the numerical integration of the equation of motion:

$$u_i^{n+1}=u_i^n+\Delta t\left(f_i^n-\tilde{\nabla}\big|_i P^n\right) \qquad \text{(Eq.2)}$$

With given right hand side and given boundary conditions, Eq.1 is a linear system of algebraic equations. Let 's suppose, we solve this for $P^n$, then we update the velocity field by using Eq.2. Now we check the continuity for the new velocity field:

$$\tilde{\nabla}\cdot u_i^{n+1}=\tilde{\nabla}\cdot u_i^n+\Delta t\left(\tilde{\nabla}\cdot f_i^n-\tilde{\Delta}P^n\right)$$

Is the new velocity field divergence free?

**≈0 only approximate solution is possible!**

No. The numerical error of the solution of the Poisson equation will be accumulated in the continuity equation.

## The accumulation of numerical errors can be avoided

Not the Poisson equation is important, but the continuity.

Instead of the original Poisson equation we solve the following:

$$\tilde{\Delta} P^n = \tilde{\nabla}\cdot f_i^n + \frac{1}{\Delta t}\tilde{\nabla}\cdot u_i^n$$

then we update the velocity by using the eq. of motion:

$$u_i^{n+1} = u_i^n + \Delta t\left(f_i^n - \tilde{\nabla}\big|_i P^n\right)$$

Now we check the divergence of the new velocity field:

$$\tilde{\nabla}\cdot u_i^{n+1} = \Delta t\left[\underbrace{\frac{1}{\Delta t}\tilde{\nabla}\cdot u_i^n + \tilde{\nabla}\cdot f_i^n - \tilde{\Delta} P^n}_{\approx 0}\right]$$

**≈0 this is what we solve for $P^n$.**

The error of continuity equation is limited by the error of solution of the discrete Poisson equation in the last time step.

## Projection method

The same method with different notations:

1-st step we evaluate: $u^*_i = u_i^n + \Delta t\, f_i^n$ u*: pseudo-velocity

2-nd step **we solve**: $\tilde{\Delta} P^n = \frac{1}{\Delta t}\tilde{\nabla}\cdot u^*_i \longrightarrow P^n$

3-rd step we evaluate: $u_i^{n+1} = u^*_i - \Delta t\,\tilde{\nabla}\big|_i P^n$

Let's check it! $\tilde{\nabla}\cdot u_i^{n+1} = \Delta t\left[\underbrace{\frac{1}{\Delta t}\tilde{\nabla}\cdot u^*_i - \tilde{\Delta} P^n}\right]$

**this (=0) is solved in the 2-nd step!**

## Steady lows

- **Small time steps required:**
  Due to the explicit method used for time integration the method is only conditionally stable.
  If the solution changes slowly, or steady, we need high amount of time steps.
- **Missing time derivative:**
  Actually in steady flow there are no time derivatives in the governing equations.

## P-u iteration for steady flow (1)

We want to fulfill in step n+1 the following with the highest possible accuracy:

$$A_P u_{i,P}^{n+1} + \sum A_\ell u_{i,\ell}^{n+1} = Q_i - \tilde{\nabla}\big|_i P^{n+1} \qquad \text{és} \qquad \tilde{\nabla}\cdot u_i^{n+1} = 0$$

Only the old pressure value can be used... (the continuity is not accurately fulfilled in this stage)
$u^n$ is used as an initial value for u*.

$$A_P u^*_{i,P} + \sum A_\ell u^*_{i,\ell} = Q_i - \tilde{\nabla}\big|_i P^n$$

**1-st step** → $u_i^*$

$$u^*_{i,P} = \frac{Q_i - \sum A_\ell u^*_{i,\ell}}{A_P} - \frac{1}{A_P}\tilde{\nabla}\big|_i P^n$$

$$u^*_{i,P} = \breve{u}_{i,P} - \frac{1}{A_P}\tilde{\nabla}\big|_i P^n$$

$u^{n+1}$ is calculated from an approximate formula from the new pressure field:

$$u_{i,P}^{n+1} \approx \breve{u}_{i,P} - \frac{1}{A_P}\tilde{\nabla}\big|_i P^{n+1}$$

**3-rd step**

$u^{n+1}$ must fulfill the continuity!
Let's take the numerical divergence:

$$\tilde{\Delta} P^{n+1} = A_P \tilde{\nabla}\cdot\breve{u}_i$$

**2-nd step** → $P^{n+1}$

**Due to using old values for the neighboring velocities in the 3-rd step this is not fully accurate. We need to iterate.**

## P-u iteration for steady flow (2)

- **Inner iteration**:
  Iterative solution methods are used for solving the algebraic systems in 1-st and in 2-nd step. Unusually only 1 inner iteration step is done.
- **Pressure equation:**
  The Poisson equation is solved for pressure correction (not for pressure). This reduces the round-off error.
- **SIMPLE, SIMPLEC, SIMPLER, PISO**
- **Time dependent models:**
  When modeling transient flows we can include the time derivatives into Q. This way, the application of implicit integration scheme is possible, which allows much larger time steps.