2. Irrotational flows

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Irrotational flows

Shape of the streamlines?
Pressure and velocity distributions?

Most analytic solutions have been developed for irrotational flows.
Lamb, H: Hydrodynamics, 1932. (First edition: 1879.)

Flows originated from a volume containing fluid at rest is an irrotational flow until the vorticity generated by walls penetrates the flow field.

„The irrotational motion of a liquid occupying a simply-connected region has less kinetic energy than any other motion consistent with the same normal motion of the boundary." (W.Thomson, 1849)

If the velocity field is rotation free: $\nabla \times \vec{v} = 0$

we can define velocity-potential function ϕ as: $\vec{v} = \nabla \phi$

(This holds for compressible flows as well.)

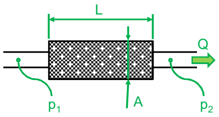
Some application examples


- Flow close to the extraction point
- Flow around airfoils
- Darcy flow, wells
- Drinking water reservoirs


Porous media flows

In porous media (such in soil, rock, or in adsorber beds) the flow can be described by Darcy's law, according to which the flow rate in a horizontal sample:

$$Q = -\frac{p_2 - p_1}{L} A \frac{k}{\mu}$$



in which, μ [Pa.s]=$\rho\nu$ is the dynamical viscosity of the fluid, k [m^2] is the permeability of the porous medium. In most cases k is measured in Darcy units: 1 D ≅ 10^{12} m^2. The generalization of Darcy's law:

$$\vec{v} = -\frac{k}{\mu}\nabla(p + \rho g z) \qquad Q = \int \vec{v}\, d\vec{A}$$

thus the velocity potential:

$$\phi = -\frac{k}{\mu}(p + \rho g z)$$

Calculation of the pressure field

The equation of motion for Darcy flow:

$$\phi = -k\frac{p + \rho g z}{\mu}$$

In which the density (ρ), the permeability (k) and the dynamic viscosity (μ) are constant. So the pressure difference reads:

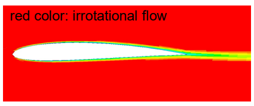
$$p_2 - p_1 = \frac{\mu}{k}(\phi_1 - \phi_2) + \rho g (z_1 - z_2)$$

Pressure distribution in ideal fluid (μ=0, ρ=const.) can be obtained from the Bernoulli principle:

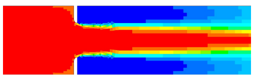
$$p_2 - p_1 = \frac{\rho}{2}\left(v_1^2 - v_2^2\right) + \rho g (z_1 - z_2)$$

Nearly irrotational flows in reality

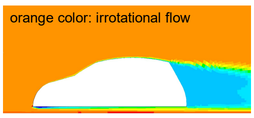
Total pressure distributions from 2D CFD models:



around an airfoil



in an orifice



around the body of a car

Velocity potential (ϕ) for constant density fluid flow

Continuity equation: $\nabla \cdot \vec{v} = 0$

$$\nabla \cdot (\nabla \phi) = \Delta \phi = 0$$

ϕ is an harmonic function (fulfilling the Laplace equation).

An important example: velocity potential of a point source:

$$\vec{v} = \frac{Q}{4r^2\pi}\vec{e}_r \longrightarrow \phi = -\frac{Q}{4\pi r} + \text{Const.}$$

Because of the linearity of the mathematical model, the solutions (ϕ functions) can be combined: **superposition principle**.

Stream function (Ψ)

The continuity equation of a **constant density** fluid is automatically fulfilled, if the velocity field can be derived from an existing Ψ vector potential function:

Def: $\vec{v} = \nabla \times \vec{\psi}$

$$\nabla \cdot \vec{v} = \nabla \cdot \nabla \times \vec{\psi} \equiv 0$$

Ψ is a scalar function in 2 spatial dimensions and called the „stream function" in 2D flow situations. Only the z component is non-zero:

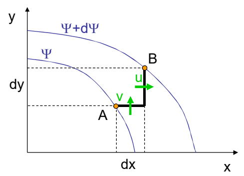
$$\vec{v} = \begin{pmatrix} \frac{\partial \psi_z}{\partial y} - \frac{\partial \psi_y}{\partial z} \\ \frac{\partial \psi_x}{\partial z} - \frac{\partial \psi_z}{\partial x} \\ \frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y} \end{pmatrix}$$

u and v are the x and y components of the velocity vector:

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

Ψ makes much more sense in 2D, because the definition decreases the number of unknown scalar fields.

The stream function in 2D



The total differential of the stream function:

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

$$\frac{\partial \psi}{\partial x} = -v \quad \text{and} \quad \frac{\partial \psi}{\partial y} = u$$

$$d\psi = -v\,dx + u\,dy$$

Ψ expresses volume flow-rate between A and B (in a 1m wide domain):

$$Q_{A-B} = \psi_B - \psi_A$$

There is no flow through the iso-lines of Ψ, therefore these are streamlines.

The continuity in 2D: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$

2D irrotational flow of a constant density fluid

Let's suppose, that:

$$\nabla \times \vec{v}\big|_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial \psi}{\partial x} = -v \quad \text{and} \quad \frac{\partial \psi}{\partial y} = u$$

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0$$

Ψ is also a harmonic function.

Complex potential (w)

Both Ψ and ϕ are harmonic functions: $\Delta \Psi = 0$ and $\Delta \phi = 0$

furthermore they fulfill the Cauchy-Riemann conditions:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Therefore they can be regarded as the real and imaginary parts of a differentiable complex function: $w = \phi + i\psi$ w is called complex potential.

$w = f(z)$ z is a complex number (position vector); $z = x + iy$

Thus, any differentiable complex function corresponds to valid 2D, steady, irrotational flow of a constant density fluid.

We only need to look for solutions fulfilling the boundary conditions.
We will analyze the complex potential of some primitive flow structures, then we superimpose and transform these simple solutions for obtaining solutions which fulfill more complex boundary conditions.

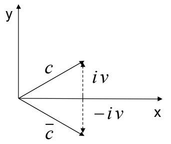
The complex velocity

Velocity is a complex vector as well:

$$c = u + i\,v$$

The complex conjugate of the velocity vector can be obtained by taking the derivative of the complex potential:

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial iy} = u - i\,v = \bar{c}$$



Potentials

	Ψ	ϕ	w
Name	Stream func.	Velocity-pot.	Complex-pot.
Definition	$\nabla\times\vec{\psi}=\vec{v}$	$\nabla\phi=\vec{v}$	$w=\phi+i\psi$
Rotational flow	applicable	N.A	N.A
3D flow	vector	scalar	N.A
Variable density flow	N.A **	applicable	N.A

** Another definition of Ψ allows compressibility.

Parallel flow

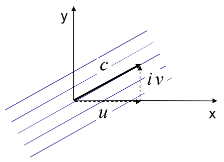
$w=\bar{c}\,z$ $\bar{c}$ is a complex number.

$$\bar{c}=\frac{dw}{dz}=u-i\,v$$

$$w=(u-i\,v)(x+i\,y)=\underbrace{ux+vy}_{\phi}+i\underbrace{(-vx+uy)}_{\psi}$$

E.g: the streamline Ψ=0 is a straight line passing through 0,0 :

$$y=\frac{v}{u}x$$

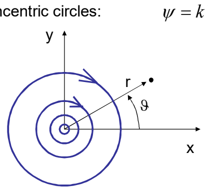


Free vortex 1.

$w=i\,k\ln z$ k is a real number.

$$w=i\,k\ln\left(r\,e^{i\vartheta}\right)=\underbrace{-k\vartheta}_{\phi}+i\underbrace{k\ln r}_{\psi}$$

Streamlines are concentric circles: $\psi=k\ln r=\text{Const.}$



Free vortex 2.

The velocity field

$$\bar{c} = \frac{dw}{dz} = i\frac{k}{z} = i\frac{k}{r e^{i\vartheta}} = i\frac{k}{r}e^{-i\vartheta}$$

$$\bar{c} = \frac{k}{r} i\left(\cos(-\vartheta) + i\sin(-\vartheta)\right)$$

$$c = \frac{k}{r}\left(\sin\vartheta - i\cos\vartheta\right)$$

Unit vector pointing in azimuthal direction.

The velocity magnitude: $|c| = \frac{k}{r}$

Circulation along any curve which passes around the origo one time:

$$\Gamma = 2r\pi|c| = 2r\pi\frac{k}{r} = 2\pi k$$ thus: $$k = \frac{\Gamma}{2\pi}$$

Problem #2.1

What is the shape of the water surface above the drain of the bath tub? Determine the drop of water level between points characterized by r_1 and r_2 for a given value of Γ!

$v_z \approx 0$ the field variables depend only on r.

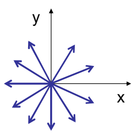
To the solution

Sources and sinks

Note that, these are line sources in 3D.

$$w = k\ln z$$

k is a real number.



$$w = k\ln\left(r e^{i\vartheta}\right) = \underbrace{k\ln r}_{\phi} + i\underbrace{k\vartheta}_{\psi}$$

$$z = x + iy \longrightarrow \psi = k\,atg\frac{y}{x}$$

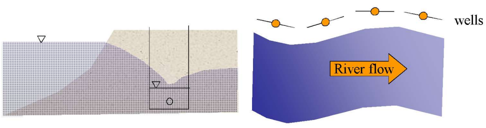
$$\bar{c} = \frac{dw}{dz} = \frac{k}{z} = \frac{k}{r}\left(\cos\vartheta - i\sin\vartheta\right)$$

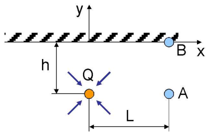
$$c = \frac{k}{r}\left(\cos\vartheta + i\sin\vartheta\right)$$

Unit vector of radial direction.

$$Q\left[\frac{m^2}{s}\right] = \psi_{\vartheta=2\pi} - \psi_{\vartheta=0} = k\,2\pi$$ therefore: $$k = \frac{Q}{2\pi}$$

Problem #2.2





a. Construct the complex potential for this flow! (Q, h and L are given.)
b. Determine the velocity magnitude in B!
c. What is the volume flow-rate between A and B?
d. Calculate the pressure distribution along axis x for Darcy flow of a given permeability and viscosity!

To the solution

Flow around a corner

$$w = \frac{k}{n} z^n$$

k, n: real numbers, and n>0.

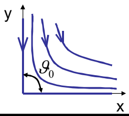
$$w = \frac{k}{n} r^n e^{in\vartheta} = \frac{k}{n} r^n \left(\cos n\vartheta + i \sin n\vartheta\right)$$

$$\psi = \frac{k}{n} r^n \sin n\vartheta$$

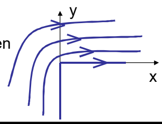
$$\psi = 0, \text{when} \quad \vartheta = 0, \frac{\pi}{n}, \frac{2\pi}{n}, \ldots$$

$$w = \frac{k\vartheta_0}{\pi} z^{\pi/\vartheta_0}$$

n=2 :
Ψ=0, when
0, π/2



n=2/3 :
Ψ=0, when
0, 3π/2



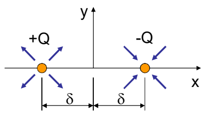
Problem #2.3

a. What is the shape of the streamlines close to a stagnation line? y=f(x)

b. How does the velocity of a fluid parcel approaching the stagnation line changes with distance? v=g(y)

To the solution

Dipoles (doublets) (1)



$$\delta \to 0,\quad Q \to \infty,\quad Q \cdot \delta = \text{const.}$$

$$w = \frac{Q}{2\pi}\left[ln(z+\delta) - ln(z-\delta)\right]$$

$$\bar{c} = \frac{Q}{2\pi}\left[\frac{1}{z+\delta} - \frac{1}{z-\delta}\right]$$

$$\bar{c} = \frac{Q}{2\pi}\frac{z-\delta-(z+\delta)}{z^2-\delta^2}$$

$$\bar{c} = -\frac{Q\delta}{\pi}\frac{1}{z^2-\delta^2} = -\frac{M}{z^2}$$

$$w = \frac{M}{z} \longleftarrow$$

Problem #2.4

a) Prove that the streamlines are circular, and touching upon the x axis from the positive y direction, in the origin of the coordinate system!
b) Please, sketch the streamlines!

To the solution

Flow around a circular cylinder (1)

$$w = c_\infty z + \frac{M}{z}$$ c_∞ is real.

$$w = c_\infty r e^{i\vartheta} + \frac{M}{r}e^{-i\vartheta} = c_\infty r(\cos\vartheta + i\sin\vartheta) + \frac{M}{r}(\cos\vartheta - i\sin\vartheta)$$

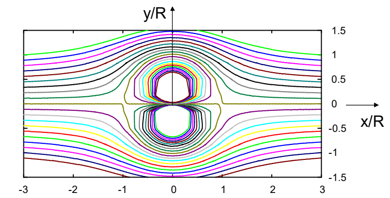
$$\Psi = \left(c_\infty r - \frac{M}{r}\right)\sin\vartheta$$

What is the equation of the streamline characterized by Ψ=0?

$\vartheta = 0$ line and the central circle of radius R, for which: $c_\infty R - \frac{M}{R} = 0$

$$\frac{M}{c_\infty} = R^2 \longrightarrow w = c_\infty\left(z + \frac{R^2}{z}\right)$$

Flow around a circular cylinder (2)



$$w = c_\infty \left(z + \frac{R^2}{z} \right) \longrightarrow \bar{c} = c_\infty \left(1 - \frac{R^2}{z^2} \right)$$

$$\bar{c}\Big|_{r=R} = c_\infty \left(1 - \frac{R^2}{R^2} e^{-2i\vartheta} \right) = c_\infty (1 - \cos 2\vartheta + i \sin 2\vartheta)$$

Flow around a circular cylinder (3)

$$\bar{c}\Big|_{r=R} = c_\infty (1 - \cos 2\vartheta + i \sin 2\vartheta)$$

$$|c|^2_{r=R} = (c\,\bar{c})_{r=R} = c_\infty^2 \left[(1 - \cos 2\vartheta)^2 + \sin^2 2\vartheta \right]$$

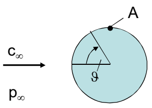
$$|c|^2_{r=R} = c_\infty^2 \left[1 - 2\cos 2\vartheta + \underbrace{\cos^2 2\vartheta + \sin^2 2\vartheta}_{1} \right]$$

$$|c|^2_{r=R} = 2c_\infty^2 [1 - \cos 2\vartheta]$$

$$|c|^2_{r=R} = 2c_\infty^2 \left[\underbrace{\cos^2 \vartheta + \sin^2 \vartheta}_{1} - \left(\cos^2 \vartheta - \sin^2 \vartheta \right) \right]$$

$$|c|^2_{r=R} = 4c_\infty^2 \sin^2 \vartheta \longrightarrow |c|_{r=R} = 2c_\infty |\sin \vartheta|$$

Problem #2.5

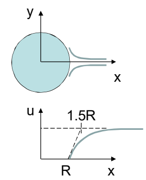


a. Calculate c_A for a given c_∞!
b. Determine the distribution of the pressure coefficient over the surface of the cylinder: $c_p = f(\vartheta)$.

To the solution

Flow around a cylinder

What is the velocity distribution in the vicinity of the stagnation point?



$$\bar{c} = c_\infty\left(1 - \frac{R^2}{z^2}\right)$$

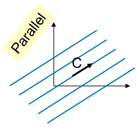
Along the x axis:

$$u = c_\infty\left(1 - \frac{R^2}{x^2}\right)$$

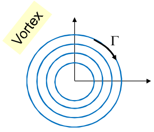
$$\frac{\partial u}{\partial x} = \frac{2c_\infty R^2}{x^3}\bigg|_{x=R} = \frac{2c_\infty}{R}$$

u relaxes to c_∞ within R/2 distance.

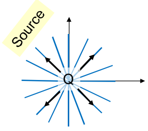
Primitive flows with physical parametrization



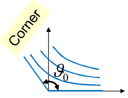
$$w = \bar{C}z$$



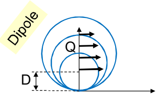
$$w = i\frac{\Gamma}{2\pi}\ln z$$



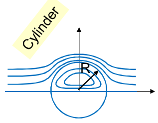
$$w = \frac{Q}{2\pi}\ln z$$



$$w = \frac{C_0\vartheta_0}{\pi} z^{\pi/\vartheta_0}$$



$$w = \frac{DQ}{z}$$

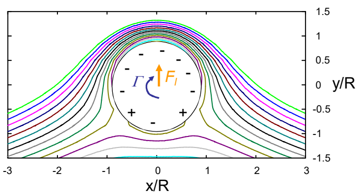


$$w = C_0\left(z + \frac{R^2}{z}\right)$$

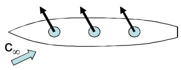
Flettner rotor (1)

$$w = \bar{c}_\infty\left(z + \frac{R^2}{z}\right) + i\frac{\Gamma}{2\pi}\ln z$$

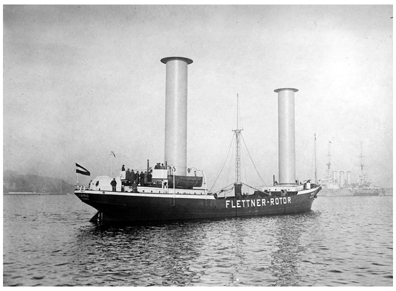
$$\frac{\Gamma}{c_\infty 2R\pi} = 1.6$$



Kutta-Joukowsky theorem: $F_l = \rho c_\infty \Gamma$



Flettner rotor (2)



[http://hu.wikipedia.org/wiki/Magnus-effektus]

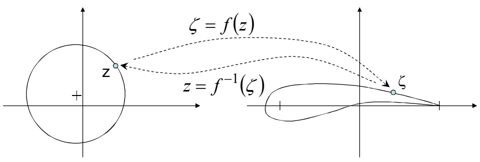
Problem #2.6

What circulation intensity is necessary for shifting the stagnation point by ϑ_0 angle?

To the solution

Joukowsky transformation (1)

We transform the z space, but we keep the value of the complex potential: $w(z)=w(\zeta)$



By using the complex potential of a Flettner rotor, we can describe the flow around an airfoil.

Joukowsky transformation (2)

A complex transformation is conformal, if it does not change the far field characteristics of the function.
These transformations can be written in the form of a series:

$$\zeta = f(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \ldots$$ in which $a_1, a_2, a_3, \ldots$ are complex numbers.

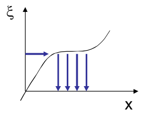
The simplest possible case is the Joukowsky transformation:

$$\zeta = z + \frac{a_{10}}{z}$$ in which a_{10} is a real number.

Singular points (1)

In those points where the derivative of the transformation expression is zero, the inverse transformation is not single valued.

A simplified illustration for a real-real transformation:



$$f'(x) = \frac{d\xi}{dx} = 0$$

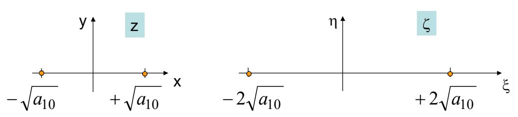
$$\zeta = z + \frac{a_{10}}{z} \longrightarrow \frac{d\zeta}{dz} = 1 - \frac{a_{10}}{z^2} = 0$$

The singular points are on the real axis in: $z = \pm\sqrt{a_{10}}$

Singular points (2)

The transformed images of the singular points:

$$\zeta = \pm\sqrt{a_{10}} + \frac{a_{10}}{\pm\sqrt{a_{10}}} = \pm 2\sqrt{a_{10}}$$



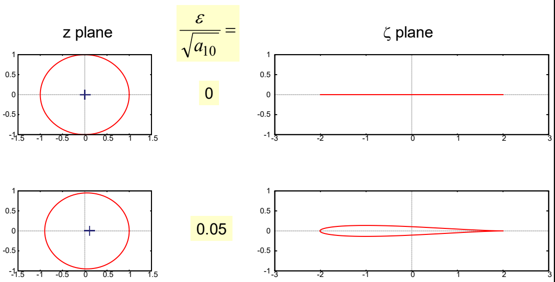
Joukowsky airfoils are images of circles passing at least through one of the singular points.

Problem #2.7

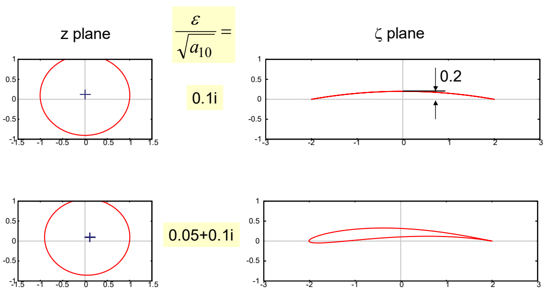
Please, specify the equation of a circle around the complex point ε, passing through the real point $\sqrt{a_{10}}$.

To the solution

Joukowsky profiles (1)

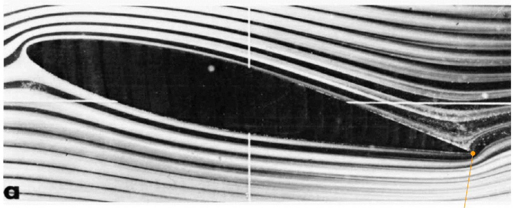


Joukowsky profiles (2)



Without circulation

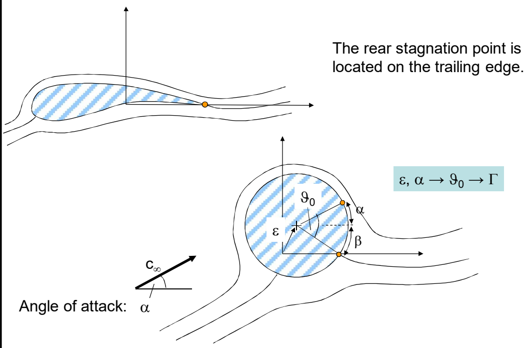
… no lift force is produced.



Hele-Show flow around an airfoil at 13° angle of attack.

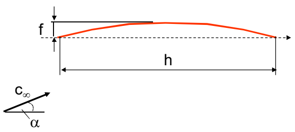
[An album of fluid motion]

Kutta condition



Problem #2.8

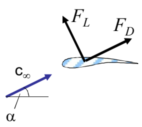
Estimate the lift coefficient (c_l) of an arched plate!
α and f/h can be regarded as given values, with both being small.



$$c_L = \frac{F_L}{\frac{\rho}{2} c_\infty^2 A}$$

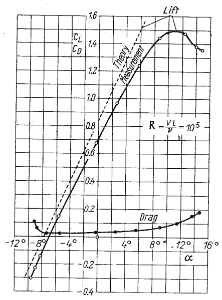
To the solution

Comparison with measured data



$$c_L = \frac{F_L}{\frac{\rho}{2} c_\infty^2 A}$$

$$c_D = \frac{F_D}{\frac{\rho}{2} c_\infty^2 A}$$



Lift and drag coefficients for a Joukowsky profile

[Schlichting 1.11]

Application examples

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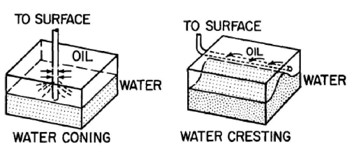
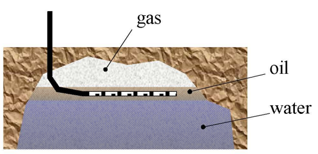
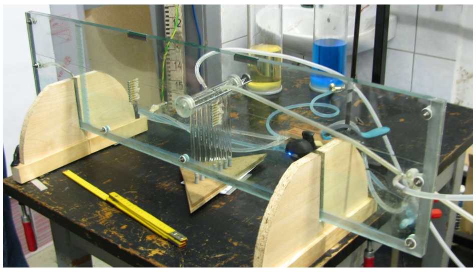
Oil wells (1)





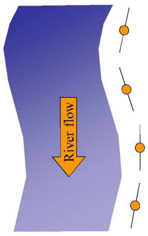
Oil wells (2)

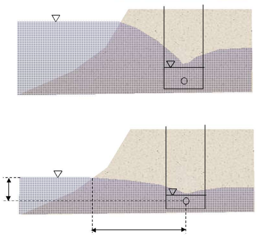
A Hele-Show experiment



[Ongoing research led by Prof. Tamás Lajos]

Onshore water wells

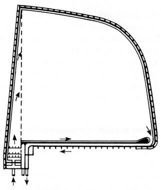




The critical flow rate depends on:
1. River level
2. The sustainable permittivity of the infiltration surface
3. Critical velocity around the pipes for avoiding damage in the porous medium.

Guber József Water Reservoir Budapest

The plans of a state of the art water reservoir operating in Munich was adapted by the Budapest water company in 1970.

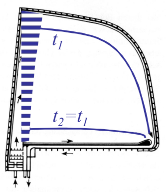


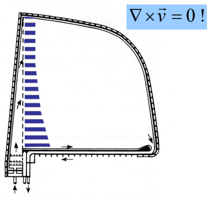
2 piano shaped reservoirs 40.000 m³ each.

Operating modes

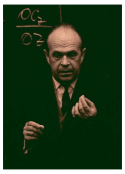
Munich
Continuous flow.
The total amount of water produced by the supplier passes through the reservoir.

Budapest
Used for network pressure stabilization.
Loaded by night, and unloaded during the peak consumption hours.



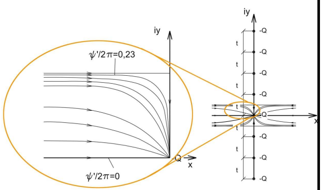


Professor József Gruber (1915-1972)



Head of Department at the Dept. Of Fluid Mechanics, BME between 1950 and 1972

Proposed the idea of irrotational flow as a design target. He also suggested a method for finding an analitical solution for the irrotational flow field.



Infinite series of sinks

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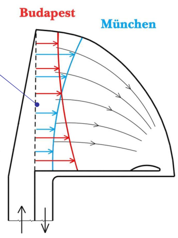
Laboratory experiments

Variable inlet comb



- Experimental setup
- Inlet comb with uniform perforation
- The Munich case
- The Budapest case

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