8. Hydraulics

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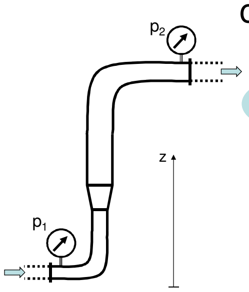
April, 2014.

Incompressible flow in closed conduits



The Bernoulli's principle:

$$p_1 + \rho g z_1 + \frac{\rho}{2} v_1^2 = p_2 + \rho g z_2 + \frac{\rho}{2} v_2^2 + \Delta p'$$

P_1 elimination of the hydrostatic pressure P_2

The total pressure loss:

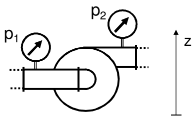
$$\Delta p' = \underbrace{\sum_i \zeta_i \frac{\rho}{2} v_i^2}_{\text{local losses}} + \underbrace{\sum_j \frac{\rho}{2} v_j^2 \frac{L_j}{d_j} \lambda_j}_{\text{losses due to straight pipe sections}}$$

Average velocity in a cross-section: $v_i = \frac{q_v}{A_i}$ in which A_i is the cross-sectional area of the pipe

Passive elements vs. pumps

We can calculate in meter dimensions:

$$\underbrace{\frac{p_1}{\rho g} + z_1 + \frac{v_1^2}{2g}}_{h_{t,1}} = \underbrace{\frac{p_2}{\rho g} + z_2 + \frac{v_2^2}{2g}}_{h_{t,2}} + \underbrace{\Delta h'}_{\Delta h' = \frac{\Delta p'}{\rho g}}$$

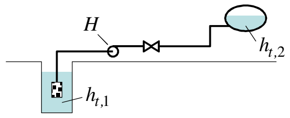


Head losses of passive elements can be restored by using pumps:

$$\frac{p_1}{\rho g} + z_1 + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + z_2 + \frac{v_2^2}{2g} - H$$

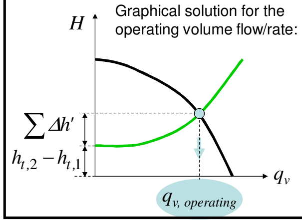
H can be regarded as the work done on unit weight of fluid. $H = f(q_v)$.

The hydraulic energy balance



$$h_{t,2} - h_{t,1} = H - \sum_i \Delta h'_i$$

Graphical solution for the operating volume flow/rate:



From this, we can express the delivery height of the pump:

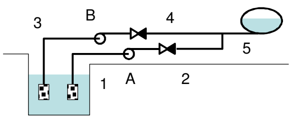
$$H = h_{t,2} - h_{t,1} + \sum_i \Delta h'_i$$

Given as: $H = f(q_v)$.

Given, constant.

Can be expressed in terms of: $q_v^2 \cdot \text{const.}$

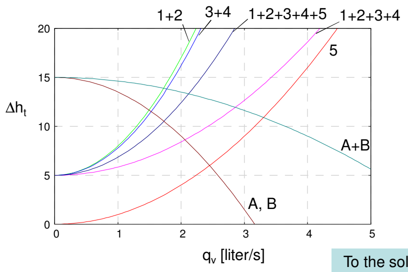
Problem #8.1



a) Explain how the characteristic curves could be combined!

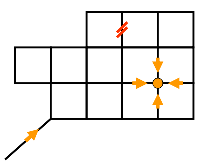
b) What is the head loss of conduit #5?

c) What total heads are produced by pump A and pump B?



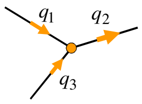
To the solution

Looped networks

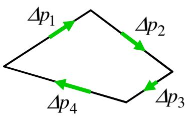


- Favorable in the cases of large supply networks (eg. in communal water supply systems).
- Water flow never stops in the conduits.
- Large local consumptions are tolerated. (Usually less pressure drop is caused.)
- When one conduit must be closed (eg. for maintenance) the rest of the supply network stays operational.

Kirchoff laws

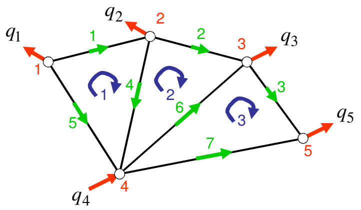


I.) The mass balance must be fulfilled in every nod.



II.) The sum of pressure drops must be zero for each loop.

Network elements



nodes: 1..N
edges: 1..E
loops: 1..L

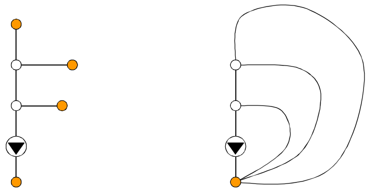
q_i represent a supply, when $q_i>0$, and consumption, when $q_i<0$.
q_i-s are localized at the nodes.
q_i values must fulfill:

$$\sum_{i=1}^{N} q_i = 0$$

Tree topology

Tree topology can always be converted into looped topology:
The nodes representing the external space are of identical pressure and must fulfill the continuity too, thus can be regarded as one single node.

E.g. the topology of an air extraction network:



The looped topology is more general than the tree topology.

Node matrix

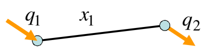
The unknowns are the x_j volume flow rates in each pipe.
+: flow direction meets the edge direction;
- : flow direction is in adverse direction.

Nodal equations:
$$q_i = \sum_{j=1}^{E} a_{ij} x_j \qquad \text{(i: 1..N)}$$

a_{ij} are the elements of the topology matrix.
a_{ij} = 1: if edge j leads out of node i;
a_{ij} = -1: if edge j leads into node i;
a_{ij}= 0: if edge j does not meet node i.

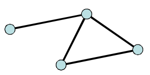
Number of equations

We have only N-1 independent nodal equation, because the sum of q_i values must be 0. Eg:



$$x_1 = q_1$$

How many nodes we have got?



$$N = 1 + E - L$$

We have E unknowns, thus:
$$E = N - 1 + L$$

Number of independent nodal equations

Number of loop equations.

With the loop equations we can close the system.

Loop equations

The total pressure loss of edge j reads:
$$\Delta p'_j = \frac{\rho}{2} \frac{x_j |x_j|}{A_j^2} \left(\frac{\ell_j}{d_j} \lambda_j + \zeta_j \right)$$

$$\Delta p'_j = k_j x_j |x_j|$$

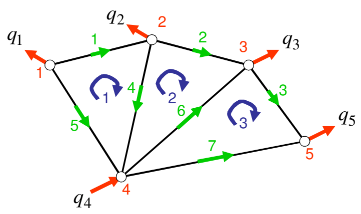
The system of loop equations is:
$$\sum_{j=1}^{E} b_{kj} \Delta p'_j = 0 \qquad \text{(k: 1..L)}$$

b_{kj} are the elements of the loop matrix:

b_{kj} are the elements of the loop matrix.
b_{kj} = 1: if the direction of edge j meets the direction of loop k;
b_{kj} = -1: if edge j is in adverse direction;
b_{kj}= 0: if edge j is not contained by loop k.

Problem #8.2

a) Specify the loop matrix for the pipe network bellow:



b) Construct the loop equation for loop 1using constant indices (1,4,5) for the unknown volume flow-rates.

To the solution

The Cross method

An easy to implement iterative solution method for looped networks.

1. Set the volume flow rates on the way to fulfill the nodal equations.
   Eg. we set $x_i=0$.
2. Correct the flow rates of all edges within loop k by adjusting their x_j values with a q_k loop correction flow rate. (Correct only one loop at a time.)
   This method does not violate the validity of the nodal equations.
3. Apply the loop corrections sequentially on each loop.
   We always spoil the pressure balance of the neighboring loops at some extents.
4. Repeat the corrections in cycles.

The loop correction (1)

The loop equations: $\sum_{j=1}^{E} b_{kj} \Delta p'_j = 0$

The corrected flow-rates must fulfill the loop equation. For loop k:

$$\sum_{j=1}^{E} b_{kj} k_j \left(x_j + b_{kj} q_k\right) \left| x_j + b_{kj} q_k \right| = 0$$

When calculating q_k we can make some approximations:

1. We can assume that the sign of x_j is not changed when being corrected:

$$\sum_{j=1}^{E} b_{kj} k_j \, sg(x_j) \left(x_j + b_{kj} q_k\right)^2 = 0$$

2. When q_k is small, its square can be neglected:

$$\sum_{j=1}^{E} b_{kj} k_j \, sg(x_j) \left(x_j^2 + 2 x_j b_{kj} q_k\right) = 0$$

The loop correction (2)

$$\sum_{j=1}^{E} b_{kj}k_j\, sg(x_j)\left(x_j^2+2x_j b_{kj}q_k\right)=0$$

$$\sum_{j=1}^{E}\left(b_{kj}k_j x_j|x_j|+2b_{kj}^2k_j|x_j|q_k\right)=0$$

q_k is a constant value within loop k, therefore:

$$\sum_{j=1}^{loop\,k} b_{kj}k_jx_j|x_j| + q_k\sum_{j=1}^{loop\,k}2b_{kj}^2k_j|x_j|=0$$

$$q_k=-\frac{\sum_{j=1}^{loop\,k} b_{kj}k_jx_j|x_j|}{\sum_{j=1}^{loop\,k}2b_{kj}^2k_j|x_j|}$$

Than we correct the flow-rates:

$$x_j^{n+1}=x_j^n+b_{kj}q_k$$

Newton-Raphson method with direct solution

Independent loop corrections do not give a convergent solution in complex cases, so we need to solve the linear system directly for the whole flow rate correction vector.

The flow rate of the j^{th} conduit is updated by taking into account q_m values of every loop:

$$x_j^{n+1}=x_j^n+\sum_{m=1}^{L}b_{mj}q_m$$

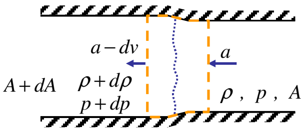
With this assumption, the loop-equation for the k^{th} loop reeds:

$$\sum_{j=1}^{E}\left(b_{kj}k_jx_j^n\left|x_j^n\right|+2b_{kj}k_j\left|x_j^n\right|\underline{\sum_{m=1}^{L}b_{mj}q_m}\right)=0$$

This formulates a system of L (k:1..L) equations for the unknown q_m (m:1..L) values can be solved by any direct solution method, e.g. by Gauss-Jordan method.

Wave propagation in long liquid product pipelines (1)

Due to the pressure jump dp, the pipe expands by dA.



Continuity:

$$(a-dv)(\rho+d\rho)(A+dA)=a\,\rho\,A \qquad a\,\rho\,dA+a\,d\rho\,A-dv\,\rho\,A=0$$

Momentum theorem:

$$A\,\rho\,a(a-(a-dv))=(A+dA)(p+dp)-Ap-\underbrace{p_{wall}\,dA}_{R}$$

Term R is the pressure force acting on the pipe wall.

$p_{wall}\approx p$ thus the Allievi theorem holds: $\rho\,a\,dv=dp$

Wave propagation in long liquid product pipelines (2)

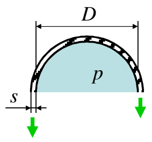
$$a\,\rho\, dA + a\, d\rho\, A - dv\,\rho\, A = 0 \quad \rightarrow \quad \frac{dv}{a} = \frac{dA}{A} + \frac{d\rho}{\rho}$$

$$\rho\, a\, dv = dp \quad \rightarrow \quad \frac{dv}{a} = \frac{dp}{\rho\, a^2}$$

$$\frac{dp}{\rho\, a^2} = \frac{dA}{A} + \frac{d\rho}{\rho}$$

$$a^2 = \frac{1}{\frac{\rho}{A}\frac{dA}{dp} + \frac{d\rho}{dp}}$$

Wave propagation in long liquid product pipelines (3)



(Hook's law) $\sigma = E_w \varepsilon$ | $\sigma = E_\ell \varepsilon$

$$\frac{dp\, D}{2s} = E_w \frac{dD}{D} = \frac{E_w}{2}\frac{dA}{A} \qquad -dp = -E_\ell \frac{d\rho}{\rho}$$

$$\frac{\rho}{A}\frac{dA}{dp} = \frac{\rho}{E_w}\frac{D}{s} \qquad \frac{d\rho}{dp} = \frac{\rho}{E_\ell}$$

$$a^2 = \frac{1}{\frac{\rho}{A}\frac{dA}{dp} + \frac{d\rho}{dp}} = \frac{1}{\frac{\rho}{E_w}\frac{D}{s} + \frac{\rho}{E_\ell}} = \frac{E_r}{\rho}$$

in which E_r is the reduced modulus:

$$\frac{1}{E_r} = \frac{1}{E_\ell} + \frac{1}{E_w}\frac{D}{s}$$

Note that, also the bubbly gas content can cause significant reduction to E_ℓ.

Problem #8.3

A) Compare the wave celerity in still water with those in a pipeline of given geometrical parameters:

Pipe diameter: 500 mm,
Wall thickness: 10 mm,
E_{water}: 2.0 x 10^9 Pa,
E_{steal}: 2.1 x 10^{11} Pa.

B) For which value of s/D ratio is the difference in sound speeds equal to 5% of the sound speed in clear water?

To the solution

Unsteady flow in liquid product pipelines

Continuity equation for constant nominal cross-section pipes:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$

The equation of motion is water-hummer equation:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + f$$

f denotes the force on unit mass of fluid due to wall friction:

$$f = \frac{1}{\rho}\frac{\Delta p'}{\Delta x}$$

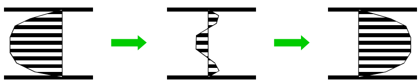
for turbulent flow, we can state:

$$\Delta p' = -\frac{\rho}{2} v|v| \frac{\Delta x}{D}\lambda \quad \text{, thus} \quad f = -\frac{\lambda}{2D} v|v|$$

Pipe friction coefficient for unsteady flows

For periodical flows of sinusoidal time dependence λ can be specified as a function of Re and St = f D / v.

When the pressure gradient changes direction:



Unsteady λ values are usually greater than the steady values due to the continuous refreshment of the boundary layer.
For laminar flow even an analytical solution can be found in the literature.

For turbulent flows λ can be identified on the basis of resonance experiments carried out in closed pipes. According to our own measurements, λ fell in the range of **0.02-0.04** (for some experiments in the ranges of Re:10^4-10^5 and St:0.005-0.02).

PDE for p(t,x) and v(t,x)

$$a^2 = \left.\frac{\partial p}{\partial \rho}\right|_{s=\text{const.}}$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

$$\underbrace{\frac{1}{a^2}\frac{\partial p}{\partial \rho}}_{1}\frac{\partial \rho}{\partial t} + \frac{v}{a^2}\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} + \rho a^2\frac{\partial v}{\partial x} = 0$$

$$\rho a\frac{\partial v}{\partial t} + \rho a v\frac{\partial v}{\partial x} = -a\frac{\partial p}{\partial x} + \rho a f$$

Now, every term is in Pa/s.

Acoustical assumptions

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} + \rho a^2 \frac{\partial v}{\partial x} = 0$$

$$\rho a \frac{\partial v}{\partial t} + \rho a v \frac{\partial v}{\partial x} = -a\frac{\partial p}{\partial x} + \rho a f$$

1) we assume: $\rho \cong \rho_0$ and $a \cong a_0$

$$\frac{\partial p}{\partial t} + \cancel{v\frac{\partial p}{\partial x}} + a_0 \frac{\partial \rho_0 a_0 v}{\partial x} = 0$$

$$\frac{\partial \rho_0 a_0 v}{\partial t} + \cancel{v\frac{\partial \rho_0 a_0 v}{\partial x}} = -a_0\frac{\partial p}{\partial x} + \rho_0 a_0 f$$

2) we assume: $v << a_0$

Since $\rho_0 a_0 v$ must be of the same order of magnitude as p .

Characteristic variables

$$\frac{\partial p}{\partial t} + a_0 \frac{\partial \rho_0 a_0 v}{\partial x} = 0 \qquad \text{(C)}$$

$$\frac{\partial \rho_0 a_0 v}{\partial t} + a_0 \frac{\partial p}{\partial x} = \rho_0 a_0 f = -\underbrace{\frac{\lambda}{2D}\rho_0 a_0}_{\zeta} v|v| \qquad \text{(M)}$$

(C+M) $$\frac{\partial}{\partial t}(p + \rho_0 a_0 v) + a_0 \frac{\partial}{\partial x}(p + \rho_0 a_0 v) = -\zeta\, v|v|$$

$$\frac{\partial \alpha}{\partial t} + a_0 \frac{\partial \alpha}{\partial x} = -\zeta\, v|v|$$ in which $\alpha = p + \rho_0 a_0 v$

(C-M) $$\frac{\partial}{\partial t}(p - \rho_0 a_0 v) - a_0 \frac{\partial}{\partial x}(p - \rho_0 a_0 v) = \zeta\, v|v|$$

$$\frac{\partial \beta}{\partial t} - a_0 \frac{\partial \beta}{\partial x} = \zeta\, v|v|$$ in which $\beta = p - \rho_0 a_0 v$

Characteristic directions

directional derivatives of α and β

$$\frac{\partial \alpha}{\partial t} + a_0 \frac{\partial \alpha}{\partial x} = -\zeta\, v|v|$$

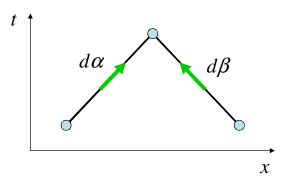
In the direction $\frac{dx}{dt} = a_0$,

$$d\alpha = -\zeta\, v|v|\, dt$$

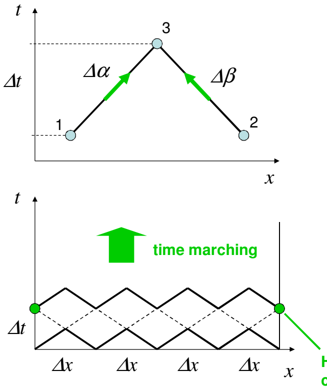
$$\frac{\partial \beta}{\partial t} - a_0 \frac{\partial \beta}{\partial x} = \zeta\, v|v|$$

In the direction $\frac{dx}{dt} = -a_0$,

$$d\beta = \zeta\, v|v|\, dt$$



Method of characteristics



Let's calculate p_3 and v_3, from given p_1, v_1 and p_2, v_2!

$$\alpha_1 = p_1 + \rho_0 a_0 v_1$$

$$\beta_2 = p_2 - \rho_0 a_0 v_2$$

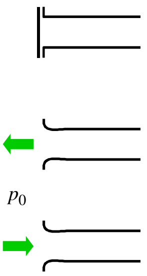
$$\alpha_3 = \alpha_1 - \zeta\, v_1 |v_1| \Delta t$$

$$\beta_3 = \beta_2 + \zeta\, v_2 |v_2| \Delta t$$

$$p_3 = \frac{\alpha_3 + \beta_3}{2}$$

$$v_3 = \frac{\alpha_3 - \beta_3}{2 \rho_0 a_0}$$

Boundary conditions



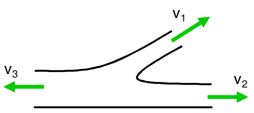
Dead end: $v = \frac{\alpha - \beta}{2 \rho_0 a_0} = 0 \rightarrow \alpha = \beta$

Outflow: $p_0 = \frac{\alpha + \beta}{2} \rightarrow \alpha = 2 p_0 - \beta$

Inflow: $p + \frac{\rho_0}{2} v^2 = p_0 \searrow$

$$\frac{\alpha + \beta}{2} + \frac{\rho_0}{2} \left(\frac{\alpha - \beta}{2 \rho_0 a_0} \right)^2 = p_0$$

Junction



By neglecting the head losses we have got the following relations:

$$v_1 A_1 + v_2 A_2 + v_3 A_3 = 0$$

$$p_1 = p_2$$

$$p_2 = p_3$$

We have got 3 incoming characteristic variables from the 3 pipes.
By using the above 3 algebraic relations we can determine the 3 unknown (outgoing) characteristics.

Problem #8.4

We suddenly open one end of an evacuated pipe.
What will be the pressure and inflow velocity immediately after the opening?
Please, use the method of characteristics and calculate α, β quantities!
Define the initial state of the pipe on the basis of v=0, p=const. conditions.

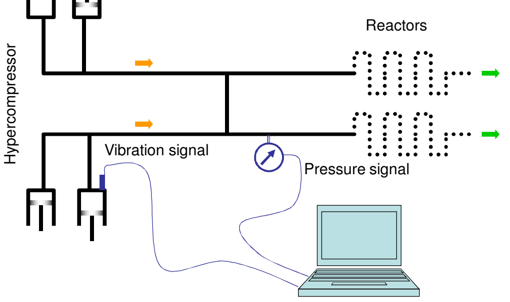
Pressure in the closed pipe:	50 kPa,
External pressure:	100 kPa,
Air density:	1.2 kg/m^3 ,
Sound speed:	334 m/s.

To the solution

Application examples

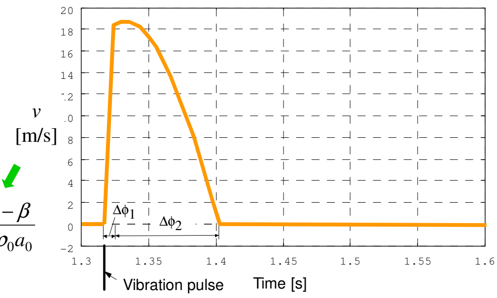
Ethylene polymerization

Operating pressure ~ 2700 bar.
Pipe stresses caused by the pressure fluctuations and by the mechanical vibrations need to be analyzed.



Boundary conditions: the compressor

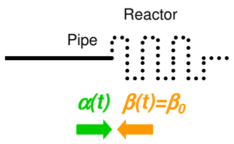
Compressor discharge (Velocity at the pipe inlet)



$$v = \frac{\alpha - \beta}{2\rho_0 a_0}$$

Phase angles of the linear ($\Delta\phi_1$) and the sinusoidal ($\Delta\phi_2$) parts are set on the basis of geometrical assumptions.
The phase angle was obtained from the vibration signal caused by the valve opening.

Boundary conditions: the reactor



Intensive dissipation due to the polymerization process.
Treated as a non-reflective BC:
a constant β value is assumed.

Simulation results vs. on site measurements

Pressure fluctuations [Pa] in quasi-periodic state