6. Gas dynamics

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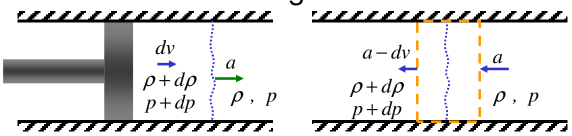
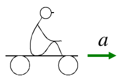
Speed of infinitesimal disturbances in still gas





Continuity:

$$A(a-dv)(\rho+d\rho)=a\,\rho\,A$$

$$a\,d\rho=\rho\,dv$$

Momentum theorem:

$$\sum \vec{I}=\sum \vec{P}$$

$$\underbrace{A\,\rho\,a}_{q_m}\underbrace{(a-(a-dv))}_{dv}=A\,dp$$

$$dp=\rho\,a\,dv$$

Allievi theorem

$$a^2=\frac{dp}{d\rho}$$

In steal	~5000 m/s
In water	~1500 m/s
In air	~340 m/s

Ideal gases

Equation of state: $\frac{p}{\rho}=RT$

We also assume that the specific heats are constant.

Internal energy: $u=c_v T$ Enthalpy: $h=u+\frac{p}{\rho}=c_p T$

Specific gas constant: $R=c_p-c_v=\frac{R_u}{M}$; $R_{air}=\frac{8314}{29}=287\left[\frac{\mathrm{J}}{\mathrm{kg\,K}}\right]$

Ratio of specific heats: $\gamma=\frac{c_p}{c_v}$ eg. for all diatomic gases:

$$\gamma=1.4$$

The speed of sound in ideal gases

We assume isentropic compression, which is very fast and the effect of the friction is negligible, thus:

$$\frac{p}{\rho^{\gamma}} = \text{const.}$$

$$ln\, p - \gamma\, ln\, \rho = ln(\text{const.})$$

$$\frac{dp}{p} - \gamma \frac{d\rho}{\rho} = 0$$

$$\frac{dp}{d\rho} = \gamma \frac{p}{\rho} = \gamma R T$$

$$a = \sqrt{\gamma R T}$$

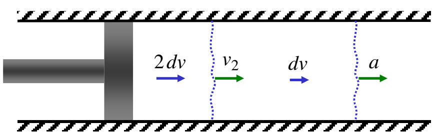
Eg. for air:

at 0°C: a=331 m/s

at 20°C: a=343 m/s

Nonlinear wave propagation

What if we generate another small disturbance?



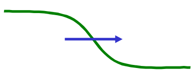
$v_2 > a$ because:

- The second wave propagates in a gas flow of dv velocity.
- The second wave propagates in a gas flow having a higher speed of sound: $p\uparrow \rightarrow T\uparrow \rightarrow a\uparrow$.

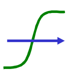
The second wave will catch up to the first wave.

Shock waves

A compression wave is steepening, and finally it becomes a **shock wave**:



Expansion waves behave in the opposite way:



- Treated as a discontinuity (finite jump) of the state variables (p, ρ, T and a).
- Propagates faster than the small disturbances. (Only shock waves can do so.)
- Deceleration of supersonic flows are generally caused by shock waves.
- It is a dissipative process. (Causes head losses.)

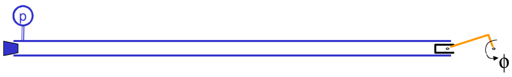
Analogy

Waves breaking in shallow water

Analogy

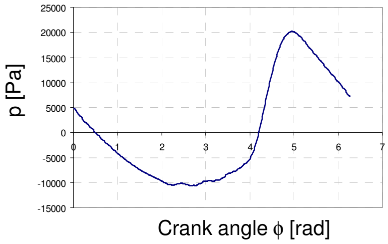
Hydraulic jump in a sink

Resonance in a closed pipe



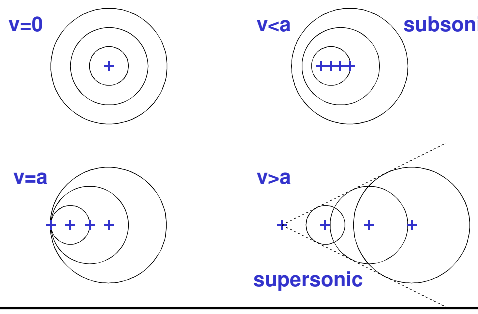
Pipe length:
6.05 m
Diameter:
36 mm
Piston displacement:
50 cm^3.

At 29 Hz we measured:



Propagation of small disturbances in subsonic and in supersonic flow

Positions of an object having velocity v at time instants 0,-1,-2 and -3 seconds and also showing the wave fronts started in those instants:

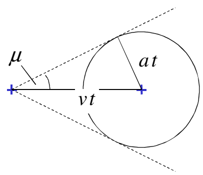


Application

Schlieren image of a gun fire

[http://www.phschool.com/science/science_news/articles/revealing_covert_actions.html]

Mach cone

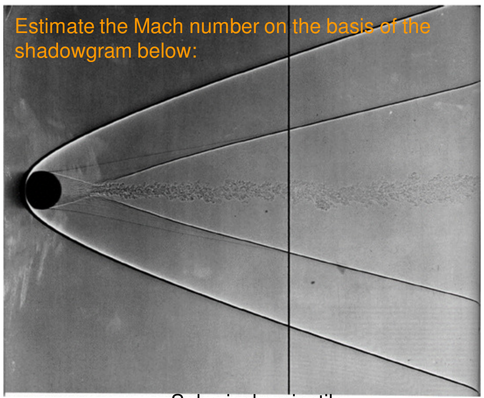


Mach number: $M = \frac{v}{a}$

Mach angle: $\mu = arc\,sin\left(\frac{a}{v}\right) = arc\,sin\left(\frac{1}{M}\right)$

Problem #6.1

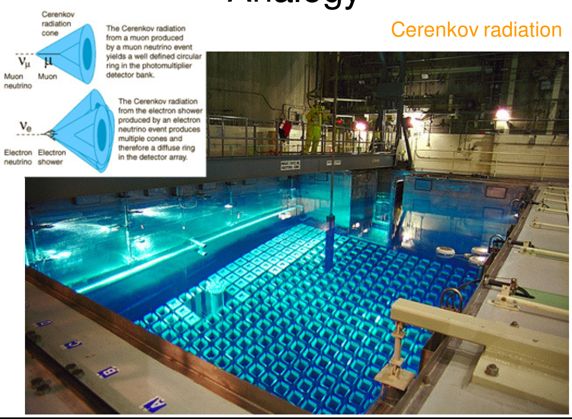
Estimate the Mach number on the basis of the shadowgram below:



[An album of fluid motion] Spherical projectile

To the solution

Analogy



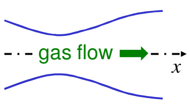
Cerenkov radiation

Variable cross-section channel (1)

Continuity: $\frac{dA}{A}+\frac{dv}{v}+\frac{d\rho}{\rho}=0$

Euler equation: $v\,dv=-\frac{dp}{\rho}$

Definition of a: $a^2=\frac{dp}{d\rho}$



A, v, p and ρ depend only on x

$$v^2\frac{dv}{v}=-\frac{dp}{\rho}\underbrace{\frac{d\rho}{dp}a^2}_{1}=-a^2\frac{d\rho}{\rho}$$

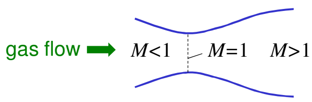
$$M^2\frac{dv}{v}=-\frac{d\rho}{\rho}=\frac{dA}{A}+\frac{dv}{v} \longrightarrow \left(M^2-1\right)\frac{dv}{v}=\frac{dA}{A}$$

Variable cross-section channel (2)

$$(M^2-1)\frac{dv}{v}=\frac{dA}{A}$$

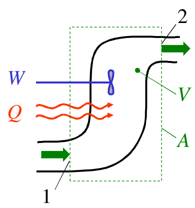
	Acceleration	Deceleration
Subsonic $M<1$	Convergent	Divergent
Supersonic $M>1$	Divergent	Convergent

If $M=1$ then $dA=0$: the area has an extreme value (minimum).



Energy equation (1)

$$\frac{\partial}{\partial t}\int_V (u+\frac{v^2}{2})\rho\, dV+\oint_A (u+\frac{v^2}{2})\rho\, \vec{v}\, d\vec{A}=Q+W-\oint_A p\, \vec{v}\, d\vec{A}$$



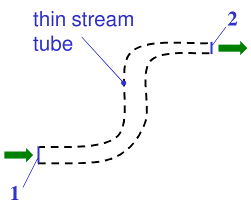
For steady state:

$$\oint_A (h+\frac{v^2}{2})\rho\, \vec{v}\, d\vec{A}=Q+W$$

Denoting the mass weighted average of the stagnation (total) enthalpy in cross-sections 1 and 2 by $h_{t,1}$ and $h_{t,2}$, it reads:

$$(h_{t,2}-h_{t,1})q_m=Q+W$$

Energy equation (2)



The stream tube can be regarded as a moving wall.

We apply the energy equation for steady flow under the following assumptions:
-the stream tube is thermally isolated (Q=0);
-the shear stress is 0 over the stream tube (W=0).

We obtain: $h_{t,2}=h_{t,1}$

Isentropic flow (1)

I. law of thermodynamics: $T\,ds = du + p\,d\left(\rho^{-1}\right)$

for an ideal gas: $T\,ds = c_v dT - \dfrac{p}{\rho^2}d\rho = c_v dT - RT\dfrac{d\rho}{\rho}$

for isentropic flow: $c_v\dfrac{dT}{T} = R\dfrac{d\rho}{\rho}$

$$\frac{R}{c_v} = \frac{c_p - c_v}{c_v} = \gamma - 1$$

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1} \longleftarrow \frac{dT}{T} = (\gamma-1)\frac{d\rho}{\rho}$$

Isentropic flow (2)

$$\frac{dT}{T} = (\gamma-1)\frac{d\rho}{\rho}$$

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

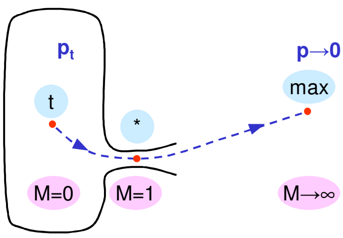
$$\frac{dT}{T} = (\gamma-1)\left[\frac{dp}{p} - \frac{dT}{T}\right]$$

$$\gamma\frac{dT}{T} = (\gamma-1)\frac{dp}{p}$$

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

Isentropic flow (3)

Reference states



Isentropic flow (4)

By applying the energy equation to a stream line we obtain:

$$h_t = h + \frac{v^2}{2} = \text{constant}$$

(It is in analogy with the Bernoulli principle.)

Relations between the reference quantities:

$$\begin{array}{ccccc} M=0 & & M=1 & & M=\infty \\ \downarrow & & \downarrow & & \downarrow \\ h_t & = & h_* + \frac{v_*^2}{2} & = & \frac{v_{max}^2}{2} \end{array}$$

$$v_* = a_*$$

Isentropic flow (5)

We can express temperature T as a function of M:

$$h_t = h + \frac{v^2}{2}$$

$$c_p T_t = c_p T + \frac{v^2}{2}$$

$$a^2 = \gamma R T = \gamma c_p \left(1 - \frac{1}{\gamma}\right) T = (\gamma - 1) c_p T$$

$$\frac{a_t^2}{\gamma - 1} = \frac{a^2}{\gamma - 1} + \frac{v^2}{2}$$

$$\frac{a_t^2}{a^2} = \frac{T_t}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

Isentropic flow (6)

Local pressure and density can be expressed in terms of the Mach number through the isentropic relations:

$$\frac{p_t}{p} = \left(\frac{T_t}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_t}{\rho} = \left(\frac{T_t}{T}\right)^{\frac{1}{\gamma-1}} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma-1}}$$

The critical ratios (for the state of M=1):

$$\frac{T_*}{T_t} = \frac{2}{\gamma + 1} \qquad \frac{p_*}{p_t} = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma-1}} \qquad \frac{\rho_*}{\rho_t} = \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma-1}}$$

For γ=1.4: **0.83** **0.53** **0.63**

Problem #6.2

Please, calculate the maximum velocity for isentropic flow if γ=1.4, R=287 J/kg-K and T_t=1000 K are given!

To the solution

Isentropic flow (8)

Mass flow-rate: $q_m = \rho v A = \frac{\rho}{\rho_t}\rho_t M \frac{a}{a_t} a_t A$

$$q_m = M\left(1+\frac{\gamma-1}{2}M^2\right)^{-\left(\frac{1}{\gamma-1}+\frac{1}{2}\right)}\rho_t a_t A$$

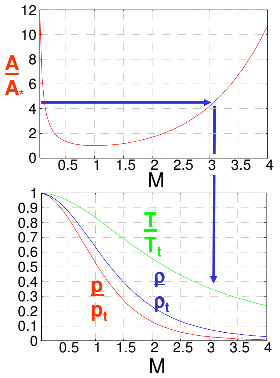
$$\frac{1}{\gamma-1}+\frac{1}{2}=\frac{2+\gamma-1}{2(\gamma-1)}=\frac{1}{2}\frac{\gamma+1}{\gamma-1}$$

$$q_m = M\left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{1}{2}\frac{\gamma+1}{\gamma-1}}\rho_t a_t A$$

$$\parallel$$

$$q_m = \left(1+\frac{\gamma-1}{2}\right)^{-\frac{1}{2}\frac{\gamma+1}{\gamma-1}}\rho_t a_t A_* \longrightarrow \frac{A}{A_*}=f(M)$$

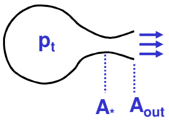
Isentropic flow (9)



$$\frac{A}{A_*}=\frac{M^{-1}\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}}{\left(1+\frac{\gamma-1}{2}\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}}$$

The inverse of the above function also gives the Mach number for a given A/A_* .

Problem #6.3



a) What is the optimum A_{out}/A_* ratio of the nozzle of a rocket thruster designed for near ground flight, if the chamber pressure p_t=10 bar$_A$, and γ=1.3. Please, use the gas tables!

b) Calculate the mass flow-rate for T_t=1300 K a, R=462 J/kg-K and A_{out}=20 cm^2!

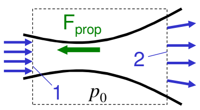
c) Please, calculate the thrust!

To the solution

Thrust function

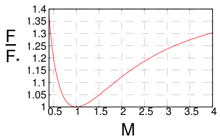
The momentum theorem for a variable cross-section steady channel flow reads:

$$F_{prop} = \left(p_2 + \rho_2 v_2^2\right)A_2 - \left(p_1 + \rho_1 v_1^2\right)A_1 + p_0\left(A_1 - A_2\right)$$



$$F = \left(p + \rho v^2\right)A$$

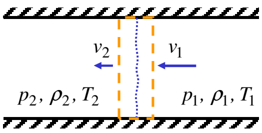
$$\frac{F}{F_*} = \frac{p + \rho v^2}{p_* + \rho_* v_*^2}\frac{A}{A_*} = \frac{p}{p_*}\frac{1+\gamma M^2}{1+\gamma}\frac{A}{A_*}$$



known functions of M. E.g:

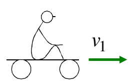
$$\frac{p}{p_*} = \frac{p_t}{p_*}\frac{p}{p_t} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}} \Bigg/ \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

Normal shock waves (1)



4 unknowns. We can eliminate one by using:

$$\frac{p_2}{\rho_2} = RT_2$$



A steady flow is observed!

Continuity: $v_1 \rho_1 A = v_2 \rho_2 A$

Momentum low: $\left(p_1 + \rho_1 v_1^2\right)A = \left(p_2 + \rho_2 v_2^2\right)A$

Energy equation: $\left(c_p T_1 + \frac{v_1^2}{2}\right)\rho_1 v_1 A = \left(c_p T_2 + \frac{v_2^2}{2}\right)\rho_2 v_2 A$

Normal shock waves (2)

Mach number was the key to isentropic flows ...

... we should try to solve this problem for $M_2(M_1)$.

$\rho_1 v_1 = ...$ → $\frac{p_1}{RT_1} M_1 (\gamma R T_1)^{1/2} = ...$

$p_1 + \rho_1 v_1^2 = ...$ → $p_1\left(1+\frac{\rho_1 v_1^2}{p_1}\right) = ...$ → $p_1\left(1+\gamma\frac{v_1^2}{a_1^2}\right) = ...$

$p_1\left(1+\gamma M_1^2\right) = ...$

$c_p T_1 + \frac{v_1^2}{2} = ...$ → $T_1\left(1+\frac{\gamma R v_1^2}{2 c_p a_1^2}\right) = ...$ → $T_1\left(1+\frac{\gamma-1}{2}M_1^2\right) = ...$

Normal shock waves (3)

(a) $\frac{p_1}{RT_1} M_1 (\gamma R T_1)^{1/2} = ...$

(b) $p_1\left(1+\gamma M_1^2\right) = ...$

(c) $T_1\left(1+\frac{\gamma-1}{2}M_1^2\right) = ...$

a*b^-1*c^0.5

$$\frac{M_1}{1+\gamma M_1^2}\sqrt{1+\frac{\gamma-1}{2}M_1^2} = \frac{M_2}{1+\gamma M_2^2}\sqrt{1+\frac{\gamma-1}{2}M_2^2}$$

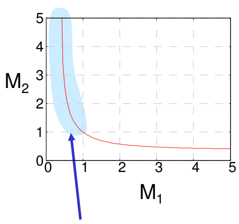
$$M_1^2\left(1+\frac{\gamma-1}{2}M_1^2\right)\left(1+\gamma M_2^2\right)^2 = M_2^2\left(1+\frac{\gamma-1}{2}M_2^2\right)\left(1+\gamma M_1^2\right)^2$$

It is a quadratic formula for M_2^2

We can arrange it into the polynomial form:

$$M_2^4(...) + M_2^2(...) + (...) = 0$$

Normal shock waves (4)



$$M_2^2 = \frac{M_1^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_1^2 - 1}$$

This branch belongs to an expansion shock.
Is it valid?

Normal shock waves (5)

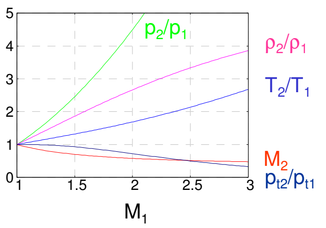
Pressure ratio: **(b)** ⟶ $$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2} = f(M_1)$$

Temperature ratio: **(c)** ⟶ $$\frac{T_2}{T_1} = \frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2} = g(M_1)$$

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1}\left(\frac{T_2}{T_1}\right)^{-1} = h(M_1)$$

Normal shock waves (6)

$$\frac{p_{t2}}{p_{t1}} = \frac{\frac{p_{t2}}{p_2}}{\frac{p_{t1}}{p_1}}\frac{p_2}{p_1} = \frac{\left(\frac{\cancel{T_{t2}}}{T_2}\right)^{\frac{\gamma}{\gamma-1}}}{\left(\frac{\cancel{T_{t1}}}{T_1}\right)^{\frac{\gamma}{\gamma-1}}}\frac{p_2}{p_1} = \left(\frac{T_1}{T_2}\right)^{\frac{\gamma}{\gamma-1}}\frac{p_2}{p_1}$$



The entropy production

The entropy change can be related to pressure and temperature ratios:

$$Tds = dh - \frac{dp}{\rho} = c_p dT - RT\frac{dp}{p}$$

$$\frac{ds}{R} = \frac{\gamma}{\gamma-1}\frac{dT}{T} - \frac{dp}{p}$$

$$\frac{s_2-s_1}{R} = \frac{\gamma}{\gamma-1}\ln\frac{T_2}{T_1} - \ln\frac{p_2}{p_1}$$

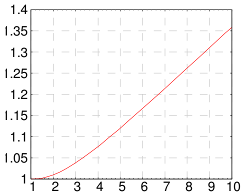
Generally we can state: $$e^{\frac{s_2-s_1}{R}} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}\frac{p_1}{p_2}$$ ⟶ For shocks: $$e^{\frac{s_2-s_1}{R}} = \frac{p_{t1}}{p_{t2}}$$

An expansion shock wave would lead to a decrease of entropy, therefore it does not exist.

Rankine-Hugoniot relations

Change of the thermodynamical state

$$\frac{\left.\frac{T_2}{T_1}\right|_{shock}}{\left.\frac{T_2}{T_1}\right|_{isent}} = \frac{\left.\frac{\rho_1}{\rho_2}\right|_{shock}}{\left.\frac{\rho_1}{\rho_2}\right|_{isent}}$$

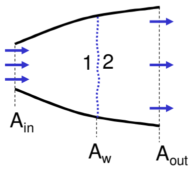


$\frac{p_2}{p_1}$

Weak shocks are almost isentropic.

... but they still propagate much faster than $\boldsymbol{a}$.

Problem #6.4



There is a strong stationary normal shock in a divergent channel at the cross-section characterized by A_w.

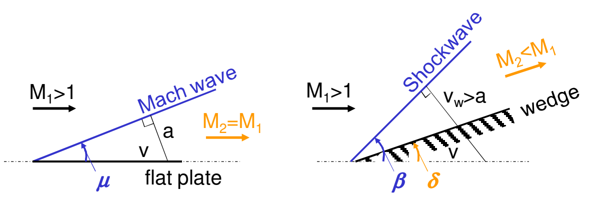
$\gamma = 1.4$	$M_{in} = 2$
$p_{in} = 100\,kPa_A$	$T_{in} = 270\,K$
$A_w / A_{in} = 2$	$A_{out} / A_{in} = 3$

a) Calculate the Mach number at the outlet (M_{out})!

b) Please, determine the outlet pressure (p_{out})!

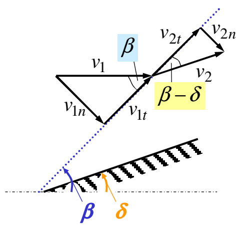
To the solution

Oblique shockwaves (1)



- Flow direction is changed by δ angle.
- In still medium, shockwaves propagate faster than the speed of sound, therefore: β>μ
- M_2 can be > 1 for an oblique shock.

Oblique shockwaves (2)



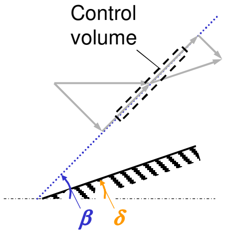
$$v_{1n} = v_1 \sin\beta$$

$$v_{1t} = v_1 \cos\beta$$

$$v_{2n} = v_2 \sin(\beta-\delta)$$

$$v_{2t} = v_2 \cos(\beta-\delta)$$

Oblique shockwaves (3)



$$\rho_1 v_{1n} = \rho_2 v_{2n}$$

$$\rho_1 v_{1n}(v_{1n} - v_{2n}) = p_2 - p_1$$

$$\rho_1 v_{1n}(v_{1t} - v_{2t}) = 0 \longrightarrow v_{1t} = v_{2t}$$

$$h_1 + \frac{1}{2}\left(v_{1n}^2 + v_{1t}^2\right) = h_2 + \frac{1}{2}\left(v_{2n}^2 + v_{2t}^2\right)$$

Same formulae are used for normal shocks!

$$\begin{cases} \rho_1 v_{1n} = \rho_2 v_{2n} \\ p_1 + \rho_1 v_{1n}^2 = p_2 + \rho_2 v_{2n}^2 \\ h_1 + \dfrac{v_{1n}^2}{2} = h_2 + \dfrac{v_{2n}^2}{2} \end{cases}$$

Oblique shockwaves (4)

We take the normal components of the Mach numbers:

$$M_{1n} = M_1 \sin\beta \qquad M_{2n} = M_2 \sin(\beta-\delta)$$

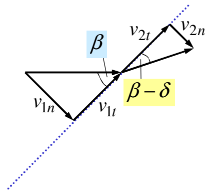
The static flow quantities can be calculated by using the gas tables developed for normal shocks:

$$M_{2n}^2 = \frac{M_{1n}^2 + \dfrac{2}{\gamma-1}}{\dfrac{2\gamma}{\gamma-1}M_{1n}^2 - 1}$$

$$\frac{p_2}{p_1} = f(M_{1n}) \qquad \frac{T_2}{T_1} = g(M_{1n}) \qquad \frac{\rho_2}{\rho_1} = h(M_{1n})$$

But the angle β is still unknown!

Oblique shockwaves (5)



$$tg\,\beta = \frac{v_{1n}}{v_{1t}} \qquad tg\,(\beta-\delta) = \frac{v_{2n}}{v_{2t}}$$

$$v_{1t} = v_{2t}$$

density ratio for a normal shock:

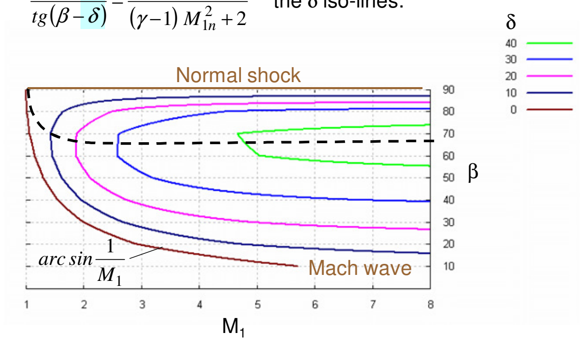
$$\frac{tg\,\beta}{tg\,(\beta-\delta)} = \frac{v_{1n}\,v_{2t}}{v_{2n}\,v_{1t}} = \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\,M_1^2 \sin^2\beta}{(\gamma-1)\,\underbrace{M_1^2 \sin^2\beta}_{M_{1n}^2}+2}$$

Now, we can plot β against M_1 for given values of δ.

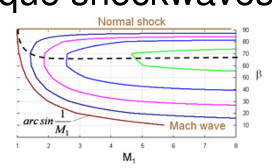
Oblique shockwaves (6)

$$\frac{tg\,\beta}{tg(\beta-\delta)} = \frac{(\gamma+1)\,M_{1n}^2}{(\gamma-1)\,M_{1n}^2+2}$$

the δ iso-lines:

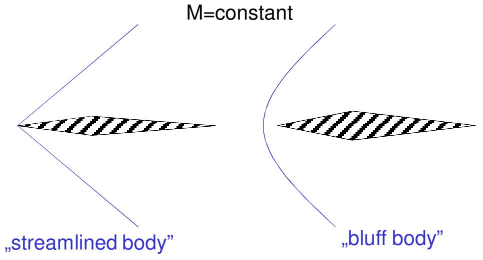


Oblique shockwaves (7)



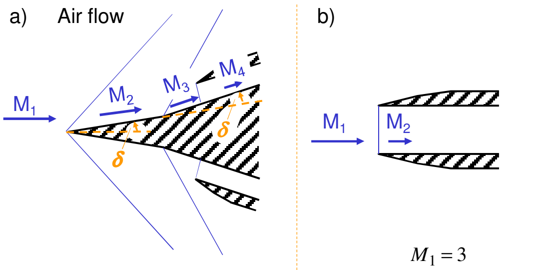
- Above a minimum Mach number M_{min} two β angles exist for a given *δ*. ($\beta_{strong} > \beta_{weak}$) Only the weak wave can be observed in external flows. (The strong wave can only be produced in wind tunnels.)
- M_{min} depends on δ. Bellow M_{min}, no oblique shock is possible. A detached bow wave is formed.
- We can also define a maximum angle δ_{max}, above which no oblique shockwave can exist for a given Mach number.

Oblique shockwaves (8)



Eg. if we increase the thickness of the wing the bow shock can be detached, the flow goes through a normal shock, therefore a we can expect a much higher pressure close to the leading edge.

Problem #6.5



a) $M_1 = 3 \quad \delta = 8°$

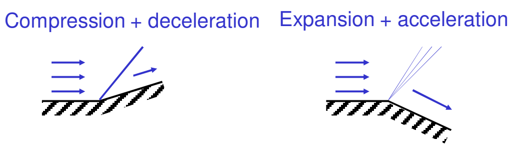
$M_2 = ? \quad M_3 = ? \quad M_4 = ? \quad \frac{p_{t4}}{p_{t1}} = ?$

b) $M_1 = 3$

$M_2 = ? \quad \frac{p_{t2}}{p_{t1}} = ?$

To the solution

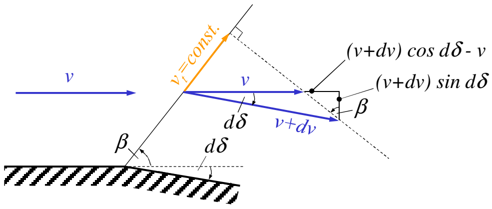
Prandtl-Meyer expansion (1)



Change of flow direction in supersonic flow (at least in isentropic cases) is directly linked to acceleration and deceleration.

We assume an isentropic process; thus we limit the analyses to expansion and to elementary compression cases.

Prandtl-Meyer expansion (2)



$$tg\,\beta = \frac{(v+dv)\cos d\delta - v}{(v+dv)\sin d\delta}$$

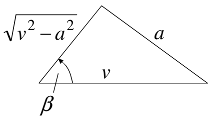
Prandtl-Meyer expansion (3)

$$tg\,\beta = \frac{(v+dv)\cos d\delta - v}{(v+dv)\sin d\delta}$$

If $d\delta \to 0$, then $\cos d\delta \to 1$, and $\sin d\delta \to d\delta$.

$$tg\,\beta = \frac{dv}{v\,d\delta}$$

β is the **Mach angle**:



$$tg\,\beta = \frac{a}{\sqrt{v^2 - a^2}} = \frac{1}{\sqrt{M^2 - 1}} = \frac{dv}{v\,d\delta} \longrightarrow d\delta = \frac{dv}{v}\sqrt{M^2 - 1}$$

Prandtl-Meyer expansion (4)

We can express dv/v in terms of the Mach number:

$$\frac{dv}{v} = \frac{dM}{M} + \frac{1}{2}\frac{dT}{T}$$

$$\frac{T_t}{T} = 1 + \frac{\gamma - 1}{2}M^2 \quad \text{in which} \quad T_t = \text{constant}$$

$$-\frac{T_t}{T^2}dT = (\gamma - 1)M\,dM$$

$$\frac{dT}{T} = -\frac{(\gamma - 1)M^2}{1 + \frac{\gamma - 1}{2}M^2}\frac{dM}{M}$$

$$\frac{dv}{v} = \frac{1 + \frac{\gamma - 1}{2}M^2 - \frac{\gamma - 1}{2}M^2}{1 + \frac{\gamma - 1}{2}M^2}\frac{dM}{M} = \frac{1}{1 + \frac{\gamma - 1}{2}M^2}\frac{dM}{M}$$

Prandtl-Meyer expansion (5)

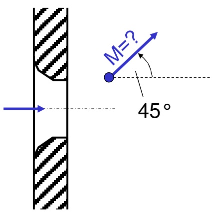
$$d\delta = \frac{dv}{v}\sqrt{M^2-1} \qquad \frac{dv}{v} = \frac{1}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

$$d\delta = \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M} \longrightarrow \delta = \int_1^M \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

This integral is the Prandtl-Meyer expansion function:

$$\delta = \sqrt{\frac{\gamma+1}{\gamma-1}}\,atg\left(\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}\right) - atg\left(\sqrt{M^2-1}\right)$$

Problem #6.6



There is a high speed air flow through a convergent nozzle. Downstream from the nozzle, at a given point, the flow direction is 45° with respect to the axis.

What is the Mach number at this point?

To the solution

Hodograph (1)

Inconveniences:
1) the length of the M vector $\rightarrow \infty$ with increasing δ angle
2) the length is not proportional to the velocity.

Therefore we will use $M^*=v/a^*$ instead of $M=v/a$:

$$M^{*2} = \frac{v^2}{a^{*2}} = \frac{v^2}{a^2}\frac{a^2}{a^{*2}} = M^2\frac{T}{T^*} = M^2\frac{T}{T_t}\frac{T_t}{T^*}$$

$$M^{*2} = M^2\left(1+\frac{\gamma-1}{2}M^2\right)^{-1}\frac{\gamma+1}{2}$$

$$M^{*2} = \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2} \quad \text{and} \quad M^2 = \frac{2M^{*2}}{\gamma+1-(\gamma-1)M^{*2}}$$

Hodograph (2)

$$d\delta = \frac{dv}{v}\sqrt{M^2 - 1} \qquad M^2 = \frac{2M^{*2}}{\gamma + 1 - (\gamma - 1)M^{*2}}$$

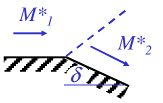
$$d\delta = \frac{dM^*}{M^*}\sqrt{\frac{M^{*2} - 1}{1 - \frac{\gamma - 1}{\gamma + 1}M^{*2}}}$$

The integral of $d\delta$ leads to the formula of an epicycloid.

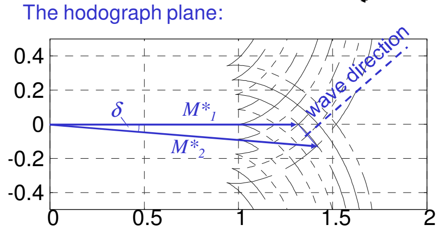
Hodograph (3)

δ and M_1 are given.
- What is the resulting M_2?
- What is the wave direction?

The physical plane:

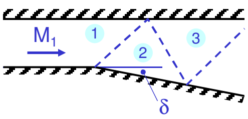


The hodograph plane:



Problem #6.7

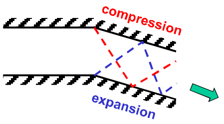
Please, solve graphically the double reflection problem below. M_1=1.28, δ=5°.



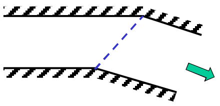
Determine M_2, M_3 and the wave directions!

To the solution

Redirection of a channel flow



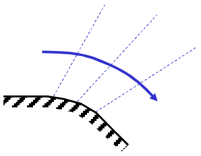
Fluctuating pressure with increased dissipation.



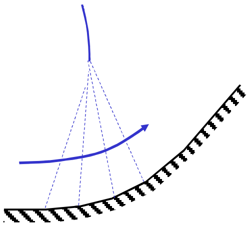
No reflected wave.
(Only one expansion wave.)

Waves past curved surfaces (1)

Expansion



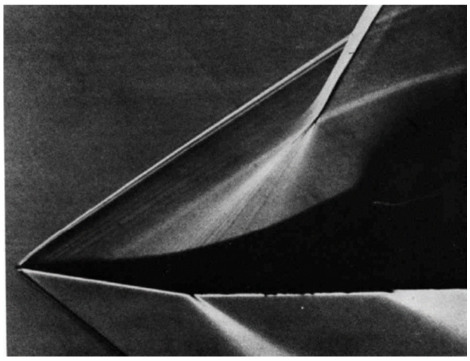
Compression



The flow is isentropic only in the near wall region.

Waves past curved surfaces (2)

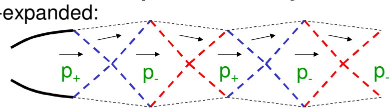
M=1.96



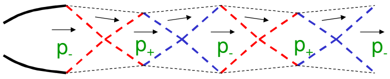
[An Album of Fluid Motion, 227]

Supersonic jets

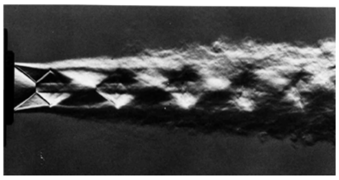
Under-expanded:



Over-expanded:

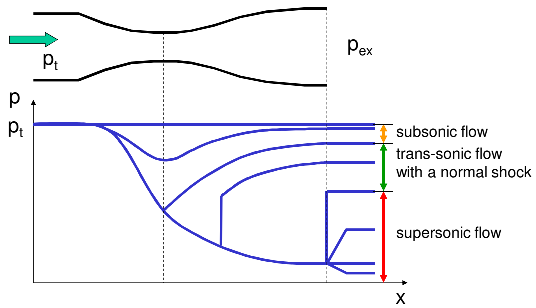


M=1.8



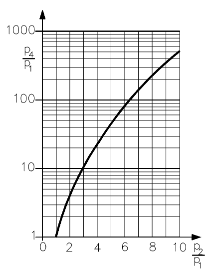
[An Album of Fluid Motion, 168]

Laval nozzle



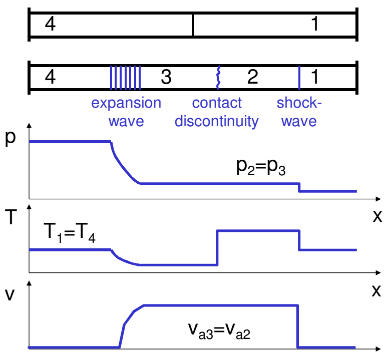
Shock tubes

An easy way to produce strong shocks or hypersonic flow.



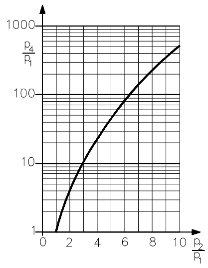
The expansion wave always has a somewhat higher pressure ratio.

The Riemann problem



Velocities in absolute frame

Problem #6.8



What is the Mach number in absolute reference frame on the upstream and downstream side of the contact discontinuity, if the initial shock tube temperature is
300 K and the initial pressure ratio is 100? (The shock tube operates with dry air.)

To the solution