Technical Acoustics and Noise Control *(lecture notes)*

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5.1. The general solution of the homogeneous wave equation, harmonic waves, (lecture notes)

The solutions of the homogeneous wave equation are the wave functions (as general solution, particular solutions, and there are solutions for free- and bounded spaces).

The general solution of the wave equation (for one-dimensional sound propagation in free space):

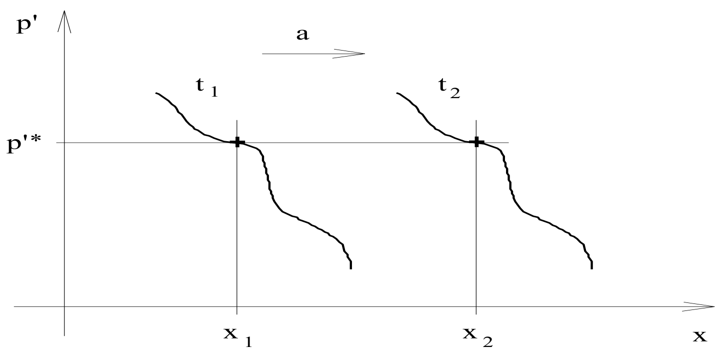
$$p'(x,t) = f\left(t - \frac{x}{a}\right) + g\left(t + \frac{x}{a}\right)$$

Comments:

- f and g functions are two times differentiable, arbitrary (optional) functions.

- The physical consequence of this "mathematic freedom", that all of the mechanical disturbances will propagate as a sound (this explain the diversity of the sound)

- The other importance of the general solution related to function argument. Let be the sound pressure in t_1 time and x_1 position p'*. Later in t_2 time the same point of the wave front (noted with a star) will move to x_2 point, see figure below.



A sound pressure distribution moving in positive x direction, in time t_1 and t_2

At plane waves propagation the sound rays are parallel to each other (there is no divergent rarefaction and convergent focus effects) and there are no attenuation and no sound generation, so

$$p'^*(x_1, t_1) = p'^*(x_2, t_2)$$

first let see the f component,

$$f\left(t_1 - \frac{x_1}{a}\right) = f\left(t_2 - \frac{x_2}{a}\right)$$

this true for an arbitrary function when arguments are the same,

$$t_1 - \frac{x_1}{a} = t_2 - \frac{x_2}{a}$$

and

$$a = \frac{x_2 - x_1}{t_2 - t_1}$$

That means, the section in the wave front noted with star, propagates in the positive x direction with the speed a. Generally the special form of the argumentum means, that the wave propagates from left to right, with the speed of sound a. The g function component describes the waves, traveling in negative x direction.

Harmonic waves:

- Harmonic mechanic excitation will create harmonic (mono chromatic, pure tone) sound wave.
- Harmonic waves can described with sine or cosine functions.
- The importance of the harmonic waves can explain as the sine and cosine functions are the base element of the harmonic (Fourier) decomposition, and the natural (free) vibration of the finite size flexible structure are harmonic vibrations (the harmonic vibration will radiate harmonic sound wave). Natural or free vibration means, that after the initial excitation the will system moves on its own, without any external interaction.
- The argument of the sine or cosine must be an angle, so any change required at argument of the general solution, and we have to introduce the phenomena phase (quantity inside the bracets right hand side) and phase state of the wave,

$$\omega\left(t - \frac{x}{a}\right) = \left(\omega t - \frac{\omega}{a}x\right) = (\omega t - kx)$$

The wave function of a $\hat{p}$ amplitude, ω angular frequency harmonic sound wave traveling in the positive x direction for the sound pressure variable is,

$$p'(x,t) = \hat{p} \cdot cos(\omega t - kx + \varphi_0)$$

Where:

Note	Measurement unit	Name	Meaning
p'(x,t)	[Pa]	sound pressure	Instantaneous pressure difference form the equilibrium value in the sound field
$\hat{p}$	[Pa]	sound pressure amplitude	the biggest pressure (magnitude) difference from the equilibrium value
$\omega t - kx + \varphi_0$	[rad]	phase angle	the position of the rotating sound pressure amplitude vector
ω=2π/T	[rad/sec]	angular frequency	the phase angle per unit time taken by the wave
T	[sec]	time of period	at x=const. place the time difference between two neighbouring phase state of the wave (e.g.: the time shift between two neighbouring positive maximum)
f=1/T	[Hz]	frequency	number of period per unit time
k=2π/λ	[rad/m]	wavenumber	the phase angle per unit length taken by the wave
λ	[m]	wave length	at t=const. time the distance between two neighbouring phase state of the wave (e.g.: the distance between two neighbouring positive maximum))

φ_0	[rad]	initial phase angle	allow to start the wave from arbitrary phase angle at t=0sec time and x=0m position

Turn back to the original form of the argument, inside the bracket the denominator of the second term is the phase velocity,

$$\omega t - kx = \omega\left(1 - \frac{x}{\omega/k}\right) = \omega\left(1 - \frac{x}{a_f}\right)\ ,\ \ a_f = \frac{\omega}{k} = \frac{2\pi/T}{2\pi/\lambda} = \frac{\lambda}{T}$$

The complex exponential representation of harmonic waves:

The complex exponential wave function of a $\hat{p}$ amplitude, ω angular frequency harmonic sound wave traveling in the positive x direction for the sound pressure variable (noted with bold p letter) is,

$$\boldsymbol{p}'(x,t) = \hat{p}\cdot cos(\omega t - kx + \varphi_0) + i\cdot\hat{p}\cdot sin(\omega t - kx + \varphi_0) = \hat{p}\cdot e^{i(\omega t - kx + \varphi_0)} =$$

$$= \hat{p}\cdot e^{i\varphi_0}\cdot e^{i(\omega t - kx)} = \hat{\boldsymbol{p}}\cdot e^{i(\omega t - kx)}$$

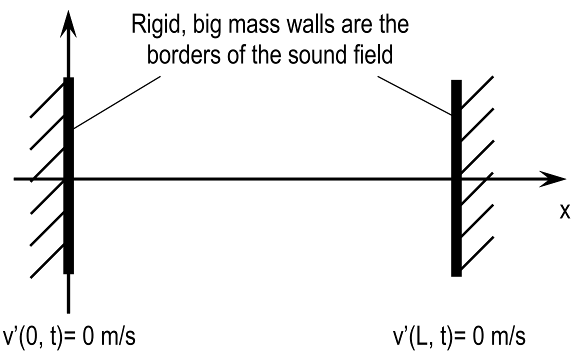
where the complex sound pressure amplitude (independent from t and x) is, $\hat{\boldsymbol{p}} = \hat{p}\cdot e^{i\varphi_0}$

The advantage of the complex exponential representation is the much more simple mathematic formalism and operation than the trigonometric one. Completed the mathematic derivations, to get the factual sound pressure we have to take the real part of the complex amount,

$$p'(x,t) = Re\big(\boldsymbol{p}'(x,t)\big)$$

6.2. The solution of the 1D homogeneous wave equation in closed space, tube resonators (lecture notes)

The general and harmonic particular solution of the wave equation related the free sound propagation. This is perfect mathematic model to solve a lot of mechanical engineering noise problem (e.g.: noise calculation of a roof fan or a road car). Another big amount of acoustic engineering exercise belongs to closed space, bordered by walls, that block the sound propagation (calculation of noise in a workshop caused by loud machines, or the modification of the sound field in a tube). To analyse the problem, we will build up a mathematic model, that allow concrete engineering design calculation, and deeper physical understanding. The first model let be one dimensional, so along the x coordinate at x= 0 m and x= L m position, perpendicular to the x coordinate two wall (with big mass and rigidity, without any holes and porosity), that is not transparent for sound is inserted.



Big mass and rigid, airtight walls block the sound propagation

The fluid mechanical no-slip condition states, that the relative velocity between the rigid body surface and the contacting fluid surface is zero. If the wall is in static rest, the neighbouring fluid particles will not move too. So the boundary conditions are,

v'(0, t)= 0 m/s   és   v'(L, t)= 0 m/s

The presence of the walls will not effect the validity of the continuity equation, the equation of motion, the energy equation and the state equation for perfect gases and the simplifications applied earlier at the linear model. So in this case the mathematic model can start from the homogeneous wave equation. Concerning the boundary conditions let change the sound pressure variable, to the particle velocity,

$$\frac{1}{a^2}\frac{\partial^2 v'}{\partial t^2}-\frac{\partial^2 v'}{\partial x^2}=0$$

The general solution,

$$v'(x,t)=f(at-x)+g(at+x)$$

Let apply the boundary conditions at x= 0 m and x= L m

$$v'(0,t)=0=f(at-0)+g(at+0) \quad \text{so,} \quad f(at)=-g(at)$$

$$v'(L,t)=0=f(at-L)+g(at+L)=-g(at-L)+g(at+L) \quad \text{so,} \quad g(at)=g(at+2L)$$

If g is periodic, the Fourie series,

$$g(at\pm x)=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}\left[\alpha_n cos\frac{2\pi n}{2L}(at\pm x)+\beta_n sin\frac{2\pi n}{2L}(at\pm x)\right]$$

The 2L is the length of the period, the wave length, so the 2π/2L ratio is the wave number

$$\frac{2\pi n}{2L}=\frac{2\pi n}{2\lambda}=k\cdot n=k_n$$

Where n= 1, 2, 3, … (natural numbers)

Let go back to the general solution of the wave equation,

$$v'(x,t)=f(at-x)+g(at+x)=-g(at-x)+g(at+x)=$$

$$=-\frac{\alpha_0}{2}-\sum_{n=1}^{\infty}[\alpha_n cosk_n(at-x)+\beta_n sink_n(at-x)]+$$

$$+\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}[\alpha_n cosk_n(at+x)+\beta_n sink_n(at+x)]$$

$$=\sum_{n=1}^{\infty}\left[\alpha_n\big(cosk_n(at+x)-cosk_n(at-x)\big)+\beta_n\big(sink_n(at+x)-sink_n(at-x)\big)\right]=$$

After the application some trigonometric identity,

$$=\sum_{n=1}^{\infty}[-2\alpha_n sink_n at\ sink_n x+2\beta_n cosk_n at\ sink_n x]$$

$$v'(x,t) = \sum_{n=1}^{\infty} [sink_n x(-2\alpha_n sin\omega_n t + 2\beta_n cos\omega_n t)]$$

Where: $k_n = \frac{2\pi}{2L} n$, and $\omega_n = \frac{2\pi}{2L} na$

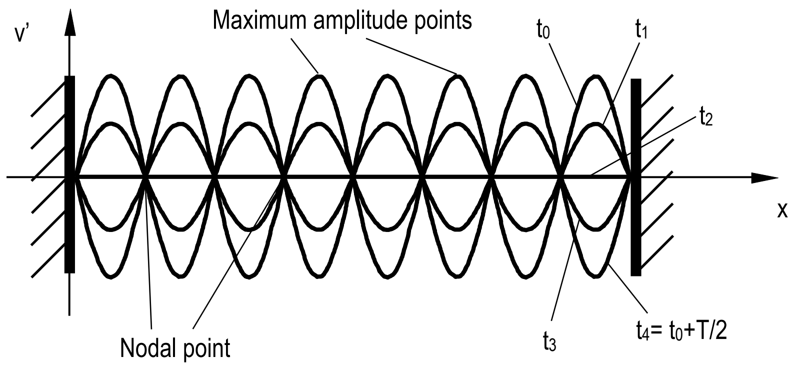
Comments:

- In the argument of the new wave function the x and t variables separate, the special (t±x/a) argumentum disappeared, so the new occurrence is not a traveling wave. (It must be true, the presence of the walls block the sound propagation!)

- The new wave function suggests that, there is a characteristic change in the physical state. To understand what is the new phenomena let take n-th element of the solution series, and let be $-2\alpha_n = \hat{v}$, and $\beta_n = 0$ than,

$$v'_n(x,t) = \hat{v}\ sin\omega_n t\ sink_n$$

The particle velocity between the walls, where $k_n x$= 0, π, 2π, 3π, … (at x= 0m and x= Lm too) independently from time, is 0m/s, these are the nodal points of the wave. Shifted with π/2 radian, where $k_n x$= π/2, 3π/2, 5π/2, … the amplitude of the wave will be maximised, these are the maximum amplitude (anti nodal) points of the wave. It is very important to understand, between two neighbouring nodal points all of the fluid particles move in the same phase, but with different amplitude. So as a consequence of the wall the propagating wave turns to the vibration of a continuous, flexible medium, that can characterise with a periodic structure of nodal and maximum amplitude point (like a vibrating spiral spring with fixed ends in walls).



The particle velocity distribution between walls at different time

- The α_n and β_n (Fourier) coefficients can determine from the v', particle velocity distribution at the starting (t= 0 sec) time, with other words from the initial condition.

- The ω_1, ω_2, ω_3, …, and the k_1, k_2, k_3, …, constants are the eigenvalues of the problem (belongs to a concrete geometric arrangement and medium). The ω_1 is called the first natural angular frequency or the base angular frequency and ω_2, ω_3, ω_4, … are the upper harmonic angular frequencies. The eigenfunctions of the problem,

$$v'_1(x,t) = sink_1 x(-2\alpha_1 sin\omega_1 t + 2\beta_1 cos\omega_1 t)$$
$$v'_2(x,t) = sink_2 x(-2\alpha_2 sin\omega_2 t + 2\beta_2 cos\omega_2 t)$$
$$v'_3(x,t) = \cdots$$

satisfying the wave equation and the boundary conditions together.

- It is very important to understand, that the eigenfunctions are possibilities for the vibrating system. The fact wat will happen in the bordered sound field fundamentally depends on the excitation. If somebody is speaking in a room, we will hear the sound of the speaker. The natural vibrations described by eigenfunctions can hear after a sound impulse excitation (e.g.: somebody clap his hand one time). Similarly when a guitar string is twanged. The natural vibrations have big importance in the engineering work, when the excitation frequency coincides one natural frequency of system, resonance will appear. All of the resonant behaviour can characterise with small amplitude excitation and big amplitude answer. The big amplitude answer in acoustics will result disturbing noisy effects, but generally an unwanted amplification of the system behaviour, so in the engineering practise usually we do not like and try to avoid resonance.

- Let be the eigenfunction of a vibrating system for the particle velocity,

$$v'(x,t) = \hat{v}\, sin\omega t\ sinkx$$

The eigenfunction for the sound pressure variables can derive from the equation of motion,

$$p'(x,t) = -\rho_0 \int \frac{\partial v'}{\partial t} dx = -\rho_0 \int \frac{\partial}{\partial t}(\hat{v}\, sin\omega t\ sinkx)dx = \rho_0 \frac{\omega}{k} \hat{v}\, cos\omega t\ coskx =$$

$$= \rho_0 a \hat{v}\, sin\left(\omega t + \frac{\pi}{2}\right) sin\left(kx + \frac{\pi}{2}\right) = \hat{p}\, sin\left(\omega t + \frac{\pi}{2}\right) sin\left(kx + \frac{\pi}{2}\right)$$

The relation between the sound pressure and particle velocity amplitudes are $\hat{p} = \rho_0 a \hat{v}$ (well known from the linear algebraic model), and between the sound pressure and particle velocity a quarter period, π/2 radian phase shift is inserted (similar, than the mass and spring one degree of freedom mechanic vibrating system).

6.3. Test questions and solved problems

T.Q.1. Write the mathematic formulation and explain the physical meaning of the general solution of the homogeneous wave equation!

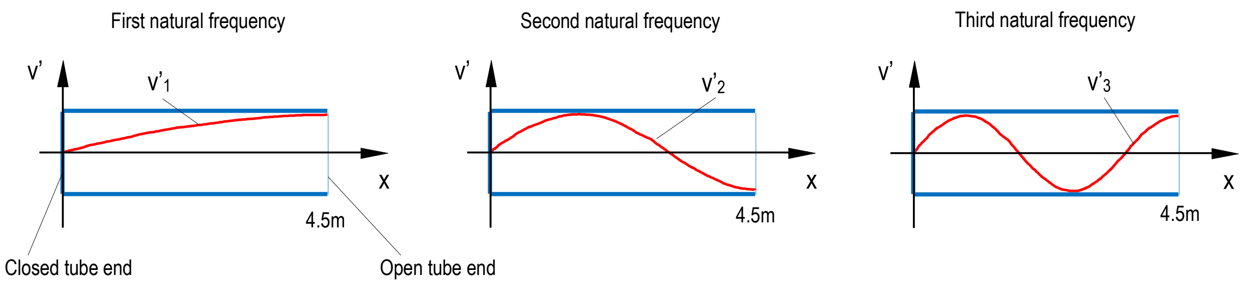
T.Q.2. Put down the wave function of harmonic waves, in trigonometric and complex representation, and explain their importance!

T.Q.3. Explain in details the solution of the 1D homogeneous wave equation in closed space, and the tube resonators!

S.P.1. Let calculate the first and third natural frequency of a 4.5m long tube, when one end is open, and the other and is blocked! The inside air temperature is 25°C.

Solution:
The speed of sound is: $a = \sqrt{\kappa R T_0} = \sqrt{1{,}4 \cdot 287 \cdot (273 + 25)} \approx 346\ m/s$

The effect inside the tube is a free harmonic continuum vibration, described by sine or cosine functions. The sine or cosine sections, satisfying the boundary conditions, can see below.



The particle velocity distribution along a one end is open, and other and is blocked tube, at the first, second and third natural frequency

The first natural (base) frequency: based on the boundary condition the particle velocity distribution is a quarter sinus from a minimum to the next minimum

$$\lambda_1 = 4 \cdot l/1 = 4 \cdot 4{,}5 = 18\ m \qquad f_1 = a/\lambda_1 \approx 346/18 \approx 19{,}2\ Hz$$

The third natural (second upper harmonic) frequency: based on the boundary condition the particle velocity distribution is a one and quarter sinus from a minimum to the second maximum:

$$\lambda_3 = 4 \cdot l/5 = 4 \cdot 4{,}5/5 = 3{,}6\ m\,, \qquad f_3 = a/\lambda_3 \approx 346/3{,}6 \approx 96{,}1\ Hz$$
