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Lecture note on

A FE Method for the Computational Fluid Dynamics of Turbomachinery

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Acknowledgements

1. Introduction

The finite element method (FEM) is an approximate method well known to the engineering community for the solution of partial differential equations (PDE) governing boundary and initial value problem (Hughes, 1987a) (Chung, 1978). Such a method transforms continuum problems into systems of algebraic equations proceeding *via*:

- the use of variational principles or weighted residual methods to transform PDE into integral equations;
- the sub-division of computational domain into many small elements of convenient shapes so that a basis for the variable interpolation could be defined.

The residual integral FE formulation is able to easily handle complex geometries and derivative type (Neumann) boundary conditions due to its intrinsic properties of working of unstructured domain discretization.

The FEM originally developed in the 1950's for aircraft structural analysis has been successfully applied to the field of non-structural problem such as fluid flows and electromagnetism starting from the work of Zienkiewicz (1965). The close relationship between finite element analysis and classical mathematical approaches, such the calculus of variations or the weighted residual methods, has established the FEM as an important branch of approximation theory.

Variational principles, used in the Rayleigh-Ritz method, unfortunately cannot always be found in fluid problems, particularly when the differential equations are not self-adjoint. Thus the weighted residual method is often applied since it requires no variational principles. Such a method utilizes a concept of orthogonal projection of a residual of a differential equation onto a subspace spanned by certain weighting functions. In the finite element method, we may use either variational principles when they exist, or weighted residuals through approximations.

The weighted residuals formulation is based on the characterization of two sets of functional spaces. The first is to be composed of the approximation or trial functions, while the second collection has to contain the variations or weighting functions. Although there are several ways of choosing such classes of functions, in finite element applications to fluid dynamics the Galerkin or Bubnov-Galerkin method, where the introduced functional spaces are composed by identical basis functions, is considered the most convenient tool for formulating FE models.

Two main instabilities origins could be detected in case of a straightforward application of the Galerkin FEM in the field of fluid dynamics of incompressible flows. Such instabilities, directly related to the mathematical character of modeling equations, could be in principle controlled by choosing proper finite element discretizations (in terms of both the mesh refinement and the pair of primitive variables interpolations). Often the dynamic response of incompressible turbulent flows (e.g. typical of rotating and stationary flow phenomena in turbomachines) is controlled by strong advective and/or diffusive mechanisms, that in practice need prohibitive level of discretization if an accurate solution has to be achieved. That is some kind of stabilization is mandatory, and the present work will be focused at the discussion of consistent weighted residual methods of achieving accurate and stable solutions.

The first instability origin is linked to the lack of diffusive term in the continuity equation so that the advective-diffusive Navier-Stokes system of equations is incomplete parabolic. As a matter of fact the incompressibility condition leads to an indetermination of the system of governing equations because the unknown pressures have to be computed out of the continuity equation, which plays the role of an additional constraint to the velocity field. From a numerical point of view, the solution to such an indetermination gives raise to the 'mixed method' where different functional spaces should be chosen for the velocity and pressure interpolation so that the satisfaction of a stability condition (known as the Babuska-Brezzi condition) is enforced. Although numerous convergent combinations of velocity and pressure 'elements' have been defined (e.g quadratic-velocity and linear-pressure), it is fair to note that in general they are not attractive from an implementational standpoint particularly for three-dimensional computations.

Besides the pressure instability in case of diffusion-dominated flows, there is a second one origin typical of the advection-dominated flows directly related to the numerical approximation of flow phenomena. In other words, the modeling of non-symmetric advective terms employing symmetric operators (such as centered finite differences or Galerkin basis function) leads to velocity solution corrupted by spurious oscillations 'wiggles'. The magnitude of oscillations is related to the convection intensity (high Reynolds or Peclet numbers) or to the presence of downstream boundary conditions forcing rapid change of flow behavior. The only way to eliminate the wiggles without changing the residual formulation requires a severe refinement of computational mesh in regions where strong gradients occur (e.g. boundary or shear layers), such that locally the flow is governed only by diffusion. As a consequence the computational load (CPU time and storage requirement) rises dramatically.

The limits implicit in the classical remedial strategies for the elimination of incompressible flow numerical instabilities, have provided the motivations for development of an alternative to the Galerkin formulation. In recent years the Petrov-Galerkin weighted residual formulations have been developed as devices for the enhancement of stability without upsetting of consistency. The leading idea is to stabilized an original Galerkin formulations adding balancing terms that emanate from a perturbation of weighting functions, giving rise to Petrov-Galerkin formulations able to circumvent the Babuska-Brezzi condition (Tezduyar et al.,1992) (Hansbo, 1995) (Hughes et al., 1986) or to introduce streamwise artificial diffusivity (Hughes et al., 1979) (Hughes, 1987b).

The present work is aimed at presenting the stabilization strategy implemented in the framework of the in-house made finite element based Navier-Stokes solver XENIOS (Rispoli and Siciliani, 1994), (Corsini, 1996), (Borello et al., 1997b) and developed on the basis of already mentioned consistent Petrov-Galerkin approaches.

The note that follows is first briefly introduced the readers to the integral methods with particular emphasis to the weighted residuals methods (Chapter 2 and 3). Then, in Chapter 4 and 5, are respectively discussed the finite element discretization technique and some basic information concerning the finite element spaces adopted within the code XENIOS. Finally in Chapter 6, the stabilized formulations for both advective and diffusive flow limits are presented and details concerning the Petrov-Galerkin stabilized of Navier-Stokes problem for turbulent and incompressible flow are also discussed.

2.   An introduction to the integral methods

Boundary value problems defined by system of PDE could be expressed in an equivalent integral form mainly on the basis of two approaches.

The first is the variational methods which transform the original differential problem introducing an integral functional **dl** (called variational principle) constructed as the inner product of the PDE onto a variation of the unknown variable. **dl** then could be considered as a virtual work, and the integral form of the problem simply claims a minimum or stationary energy condition. Such method, one of the most powerful methods of solution for engineering problems, necessarily begins with finding the variational principle.
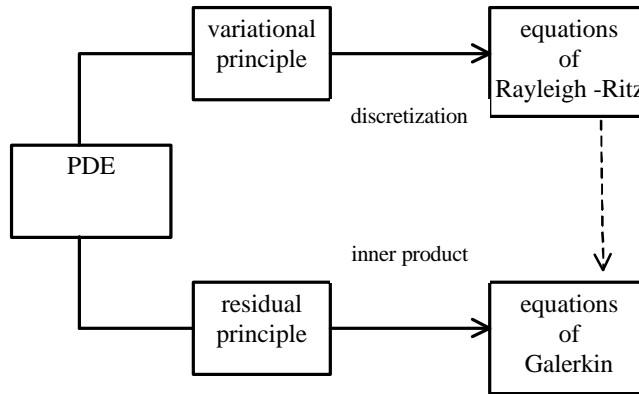

Fig. 2.1 - Variational methods versus weighted residual methods

Generally the physical law that models the dynamic of viscous flow could not be transformed in a variational principle, contrary to phenomena such as Stokes or potential flow. For such a reason the finite element formulation of Navier-Stokes problem relies on the use of the second integral approach, that is the weighted residuals method in its Galerkin definition. The weighted residuals method defines a more universal approach with respect to the variational one, because the definition of the integral problem is obtained requiring that the inner product of the residuals of the PDE to a subspace of weights is equal to zero. Such orthogonality condition is equivalent to forcing the error of the approximate differential equation to be zero in an average sense (just a finite conditions of orthogonality could be imposed) (Chung, 1978).

Let recall that the inner product between functions is equivalent to the scalar product of vectors in a Cartesian frame. The inner product of orthogonal sets of functions is therefore defined as:

$$\left(\boldsymbol{f_n}, \boldsymbol{f_k}\right) = \int_W \boldsymbol{f_n} \cdot \boldsymbol{f_k} \, dW = \boldsymbol{d_{nk}} \tag{2.1}$$

where $\boldsymbol{d_{nk}}$ is the delta of Kronecker.

Let consider now a generic differential equation of the form:

$$Lu - f = 0 \text{ in } \boldsymbol{W} \in R^{nsd}, \tag{2.2}$$

subject to the following boundary conditions:

Dirichelet conditions	u	$= g$	on $\boldsymbol{G}_g$	(2.3.a)
von Neumann conditions	$u_{,n}$	$= h$	on $\boldsymbol{G}_h$	(2.3.b)

where *nsd* defining the number of space dimensions and:

L,   is the linear differential operator,

u,   is the unknown variable,

f,   is the source term.

The boundary G of computational domain is considered as the composition of two subsets G_g and G_h, defined as follows:

$$\overline{G_g \cup G_h} = G$$
$$G_g \cap G_h = \varnothing$$

(2.4)

where the symbol $\varnothing$ defines the empty set.

The definition of a residual formulation involves the introduction of two classes of functions.

The first class S is introduced to define an approximation of the solution and it is composed by the candidate trial or trial functions u, that must satisfy the essential set of boundary conditions (2.3.a).

The second class W contains the variations or weighting functions w, used for constructing the residual orthogonal projections. This collection is very similar to the trial solutions with the exception that they have to satisfy the homogeneous counterpart of the Dirichelet boundary conditions (2.3.a).

The impossibility of defining over W continuous trial functions u and variations w (☜), gives rise to the construction of finite-dimensional approximation of S and W. These collection of functions are denoted by S^h and W^h, respectively. The superscript h refers to the association of the approximate function spaces to the discretization of the domain W, which is parameterized by a characteristic length scale.

The introduction of approximate trial solution $\tilde{u}$ into the (2.2) will not satisfy exactly the governing differential equation, and a residual or error appears that is proportional to the finite-dimensional discretization of the original continuous problem:

$$L\tilde{u} - f = e$$

(2.5)

The application of weighted residuals method requires, as already discussed, to construct the inner product of (2.5) to a set of weighting functions $\tilde{w}$:

$$\left(e, \tilde{w}\right) = \int_W e \cdot \tilde{w} \ \mathrm{d}W = 0$$

(2.6)

The above condition of orthogonality permits the definition of a residual formulation consistent with the original differential formulation that is able to guarantee the best approximation property (Hughes and Brooks, 1982).

☙

The numerical analysis tool of flow phenomena taking place in turbomachinery developed with the finite element code XENIOS is focused on the solution of the system of equations that models the steady-state dynamic response of incompressible turbulent fluids. For such a reason it is meaningful to briefly recall the

boundary value problem expressed for the following set of fluid properties (x_1-, x_2- and x_3- momentum components, pressure, turbulent kinetic energy k and viscous dissipation rate e). Here the used first order turbulence closure follows a two equations approach implemented both in its isotropic (Jones and Launder, 1972) and cubic non-linear (Craft et al., 1993) versions (Corsini, 1996) (Borello et al., 1997a).

Let consider the following vector form of the Navier-Stokes boundary value problem over a domain $W \in R^{nsd}$:

$$u_j u_{i,j} = s_{ij,j} + f_i, \qquad \text{momentum equations}$$
$$u_{k,k} = 0, \qquad \text{continuity equation}$$
$$\text{and } i,j,k = 1,nsd$$

(2.7)

with following set of boundary condition over $G = \overline{G_g \cup G_h}$:

$$u_i = g_i \to G_g$$
$$s_{ij} n_i = h_i \to G_h$$

(2.8)

In detail, the advective-diffusive equations modeling the conservation of turbulent variables employed in a standard eddy-viscosity approach are:

$$r u_j k_{,j} = \left[\left(m + \frac{m_t}{s_k} \right) k_{,j} \right]_{,j} + P - re$$

$$r u_j e_{,j} = \left[\left(m + \frac{m_t}{s_k} \right) e_{,j} \right]_{,j} + C_{e1} P \frac{e}{k} - r \frac{e^2}{k}$$

(2.9)

with following set of boundary condition over $G = \overline{G_g \cup G_h}$:

$$u_i \cdot n_i = 0 \to G_h$$

$$\frac{\P k}{\P n} = 0 \to G_h$$

$$e = \frac{\left(C_m^{1/2} k \right)^{3/2}}{k y_n} \to G_g$$

(2.10)

3.   Weak residual formulation of Navier-Stokes problem

To define and build an integral finite element based Navier-Stokes problem formulation a *weak-global approach* (Hughes, 1987a) is here proposed. The peculiarity of such a global approach is that the finite element formulation is obtained as a consequence of the discretization the weighted residual form of the problem over the

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[€] In order for *u* to be the exact solution of (2.2) it is necessary to impose the orthogonality of the residuals to an infinite set of projecting direction.

computational domain $\overline{W} = \sum_e \overline{W}_e \cup G$ decomposed into small elementary domain (e). In order to achieve a meaningful finite element residual formulation in view of the character of governing equations (2.9), that contain both first and second order terms, the classes of trial and weighting functions should satisfy generalized properties of continuity and integrability over the closed domain $\overline{W} = W \cup G \subset R^{nsd}$.

Such a global approach is equivalent to the *local* one suggested by Chung (1978), where the final form of the integral problem is a consequence of the composition of local weighted residual formulation applied in each elementary domain on the basis of functions $\tilde{u}$ and $\tilde{w}$ defined on W_e as:

$$\tilde{w} = \left\{ \tilde{w} / \tilde{w} \in C^0 \right\}$$

$$\tilde{u} = \left\{ \tilde{u} / \tilde{u} \in C^0 \right\}$$

(3.1)

that is a piecewise continuity condition on the discretized domain.

The definition of a weak integral formulation for the problem under exam requires the introduction of two function spaces S and W and their finite dimensional approximations. S^h is the approximate collection of trial functions defined as:

$$S^h = \left\{ \tilde{u} / \tilde{u} \in H^{1h}, \tilde{u} = g \rightarrow G_g \right\}$$

(3.2.a)

while W^h is the class of weighting functions:

$$W^h = \left\{ \tilde{w} / \tilde{w} \in H^{1h}, \tilde{w} = 0 \rightarrow G_g \right\}$$

(3.2.b)

It is worth to note that the component functions satisfy complementary sets of boundary values. Furthermore to preserve the sense of the application of Green-Gauss theorem to the momentum equation, such spaces must be composed by functions that must be derivable and with the derivative square-integrable over the global domain W.

Synthetically it is now possible to state the following suitable weak formulation of Navier-Stokes problem obtained from the general expression (2.6):

$$\int_W w_{ns}(u_j u_{i,j}) dW - \int_W w_{ns}(s_{ij})_{,i} dW = \int_W w_{ns} f_i dW$$

(3.3a)

Integrating by parts the diffusive flux integral (with second order derivatives) and by virtue of the Green-Gauss theorem, the weak form reads as:

$$\int_W w_{ns}(u_j u_{i,j}) dW + \int_W w_{ns,i} s_{ij} dW = \int_W w_{ns} f_i dW + \int_G w_h h_i dG$$

$$\int_W w_c u_{i,i} dW = 0$$

(3.3b)

$$u_i = g_i \rightarrow G_g$$

where w_{ns} indicates the class of weights applied to the momentum equations (together with the two advective-diffusive conservation equations of the turbulent variables), and w_c the weights applied to the scalar continuity equation. Finally, w_h indicates the restriction of weighting functions on the boundary fo computational domain.

The first thing to give attention is that the boundary condition $\boldsymbol{s}_{ij}\,\boldsymbol{n}_i = \boldsymbol{h}_i \rightarrow \boldsymbol{G}_h$ is not explicitly mentioned in the weak statement (3.3), but it is implied in the variational formulation that is only fullilled in an approximate sense due to the weighted residual approach. Boundary conditions of this type are referred as natural or Neumann conditions. Otherwise the Dirichlet or essential boundary conditions are exactly satisfied by the approximate solution function (3.2.a).

It is also important to evidentiate that the integral properties imposed to the classes $S^{\,h}$ and $W^{\,h}$ give sense to the integral term $\int_{\boldsymbol{W}} \boldsymbol{w}_{ns,i}\,\boldsymbol{s}_{ij}\,d\boldsymbol{W}$, where explictly appears the deviatoric stress tensor.

By virtue of the common integration domain and of the incompressibily condition, the system of integral equations (3.3) could be also transformed as:

$$\int_{\boldsymbol{W}} \boldsymbol{w}_{ns}\big(\boldsymbol{u}_j\,\boldsymbol{u}_{i,j}\big)d\boldsymbol{W} + \int_{\boldsymbol{W}} \boldsymbol{w}_{ns,i}\,\boldsymbol{s}_{ij}\,d\boldsymbol{W} - \int_{\boldsymbol{W}} \boldsymbol{w}_{ns}\,\boldsymbol{f}_i\,d\boldsymbol{W} - \int_{\boldsymbol{G}} \boldsymbol{w}_h\,\boldsymbol{h}_i\,d\boldsymbol{G} + \int_{\boldsymbol{W}} \boldsymbol{w}_c\,\boldsymbol{u}_{i,i}\,d\boldsymbol{W} = 0 \qquad (3.4)$$

where the continuity condition is added to the momentum equation as an additional constraint imposing to the velocity field a zero divergence, and its weighting function w_c plays the role of a Lagrangian multiplier in the bounded problem (3.4).

As already mentioned in the introductory Chapter, the pair of functions used to interpolate and weight velocity and pressure must in principle satisfy the Babuska-Brezzi stability condition. The mixed finite element formulation implemented in XENIOS adopts a quadratic variation for $\boldsymbol{w}_{ns}$ and linear variation for w_c.

4.   Finite element method and discretization

First step toward the finite element form of the weak formulation (3.4) is the definition of the functions that compose the space of solutions S and weights W, using an approximate representation of the continuum domain $\boldsymbol{W}$ on the basis of the information located in a suitable number of points (called nodes) within each elements.

In the finite element methodology implemented in XENIOS, the generic trial function $s \in S^{\,h}$ is build on $\overline{\boldsymbol{W}} = \boldsymbol{W} \cup \boldsymbol{G}$ with the following structure:

$$s = v + g \qquad (4.1)$$

where g defines the whole set of essential boundary conditions and v is a function defined on $\overline{\boldsymbol{W}} = \boldsymbol{W} \cup \boldsymbol{G}$.

Imposing the Dirichlet conditions to (4.1) follows:

$$s\Big|_{\overline{\boldsymbol{W}}} = v\Big|_{\overline{\boldsymbol{W}}/\boldsymbol{G}_g} + g \rightarrow v\Big|_{\boldsymbol{G}_g} = 0 \qquad (5.2)$$

thus it is possible to conclude that the approximate **basis** function v, as the fundamental component of S^h definition, satisfy the conditions stated for the weight space of existence $W^{\,h}$ (3.2.b).

As a consequence of the property (5.2) follows the possibility of chosing identical basis function in order to approximate the solution and to weight its residual, and the integral method based on such a choice of function

collection is named the Galerkin method. Such fundamental component function is called **basis** or **shape** function.

Let consider again the closed computational domain $\overline{W} = W \cup G \in R^{nsd}$, and a set of nodal points $l \subset \overline{W}$.

In the Galerkin residual method the structure adopted for the **basis** function rests on a discrete representation of domain that uses a finite number l of nodal points (that is a finite number of information) to build the ensuing polynomial structures:

$$w = \sum_{l=1}^{nodes} c_l \boldsymbol{f}_l \;\rightarrow\; \textbf{weighting functions} \qquad (4.3)$$

$$u = \sum_{l=1}^{nodes} d_l \boldsymbol{f}_l \rightarrow \textbf{trial functions} \qquad (4.4)$$

where:

c_l,   are the nodal values of weights,

d_l,   are the nodal values of unknown variables,

$\boldsymbol{f}_l$,   the basis functions defined in each nodes to interpolate the behavior of solution and of variation on the computational domain.

Each function belonging to the collections S^h and W^h could be thus approximately defined as a linear polynomial with constant nodal coefficients and interpolating shape functions $\boldsymbol{f}_l$, that fulfill the following properties, for each node $l \subset \overline{W}$:

$$\boldsymbol{f}_l\big|_l = 1 \;\rightarrow\; \boldsymbol{f}_l\big|_m = 0 \quad \forall\, m \subset [\,W\,] \neq l \qquad (4.5)$$

Fig.s 4.1 and 4.2 show, in the case of a linear approximation (linear shape function) over a 1-D closed domain $\overline{W}=[1,0]$, the form of each nodal interpolating functions and the effect of their linear combination to define on the whole domain the resulting interpolated solution or variation.
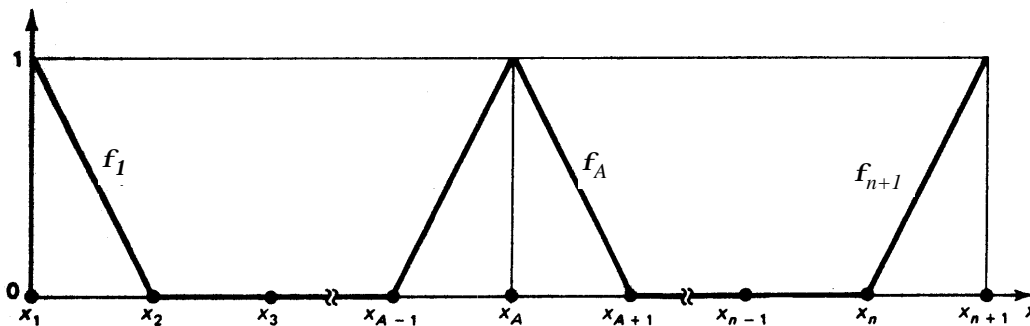

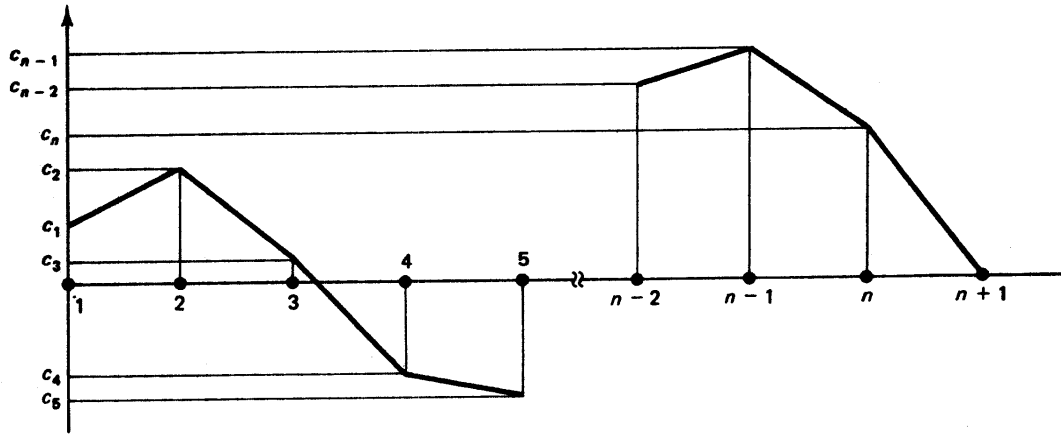**Fig. 4.1 – Nodal linear shape functions**

Fig. 4.2 – Resulting interpolated function

It is worth to note that introducing the definition of trial (4.4) and weighting (4.3) functions within the weak residual formulation (2.6) or (3.4), the set of nodal values of the variations c_l could be simplified as common constant factors on the inner product. As a consequence of such position in the discretized residual Galerkin formulation, the construction of inner product between the residuals of system of equations and the set of weights is simply obtained with reference to the nodal shape functions. That is with reference to the elementary portion of the continuum domain located by the set of nodes.

Let now consider the effect of the application of the domain subdivision into elements, shown in Fig. 4.3 and just discussed, to the weak formulation of Navier - Stokes problem (3.4).
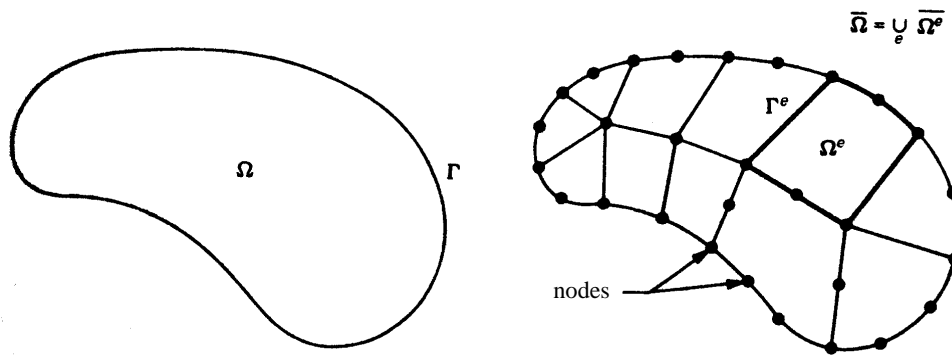
Fig. 4.3 – Computational domain finite element discretization

The domain W is shared into a finite number *NUMEL* of elementary sub-domains W_e, with $e = 1,2 \ldots,$ *NUMEL*. G_e is the boundary of the element W_e and the discretization fulfills the ensuing properties:

$$\cup_e \overline{W}_e = W \tag{4.6}$$

$$\cap_e \overline{W}_e = \varnothing \tag{4.7}$$

Finally an interior boundary could be defined as:

$$G_{int} = \cup_e G_e - G \tag{4.8}$$

Furthermore, let assume the continuity of weights and solution functions in each element W_e:

$$\tilde{w} = \left\{ \tilde{w} / \tilde{w} \in C^0 \right\} \qquad (4.9)$$

$$\tilde{u} = \left\{ \tilde{u} / \tilde{u} \in C^0 \right\}$$

whereas their first order derivatives could be discontinuous across the inter-element boundary.

The discretized formulation, with a simplified notation, reads as:

$$\sum_e \int_{W_e} w_{ns}\left(r u_j u_{i,j}\right) dW + \sum_e \int_{W_e} w_{ns,i} s_{ij} dW - \sum_e \int_{G_{int}} w_{ns}[s_{ij} n_i] dG + \sum_e \int_{W_e} w_c u_{i,i} dW = \qquad (4.10)$$

$$\sum_e \int_{W_e} w_{ns} f_i dW + \int_{G_h} w_h h_i dG$$

$$u_i = g_i \rightarrow G_g$$

where: W_e is the elementary integration domain,

G_{int} is the composition of inter-element boundaries that fall in the interior of W,

$\left[s_{ij} n_j \right] = s_{ij}^+ n_j^+ + s_{ij}^- n_j^-$, is the balance of diffusive fluxes across inter-element boundaries, their introduction in the discretized form of the diffusive integral is a consequence of the element-wise continuity characteristic of the used shape functions (4.9).

Let consider, now, the elementary domain $\overline{W}_e = W_e \cup G_e$ generically located within the domain W. The use of approximate trial and weighting functions, with an elementary domain of definition (4.9), allows the formulation of local elementwise residual formulation of Navier-Stokes problem as:

$$\int_{W_e} w_{ns}\left(r u_j u_{i,j}\right) dW - \int_{W_e} w_{ns} s_{ij,j} dW - \int_{W_e} w_{ns} f_i dW + \int_{W_e} w_c u_{i,i} dW = 0 \qquad (4.11)$$

The presence of Cauchy tensor divergence s_{ij} , including pressure and diffusive terms, leads to the application of an integration by parts and of the Green-Gauss theorem in order to lower the second order differential term and according to the local continuity properties of used basis functions.

The residual due to the stresses distribution within each element is then computed as the sum of volume contribution directly linked to the stress tensor itself and a contribution depending from the fluxes of normal and tangential stresses on the element boundary G_e:

$$\int_{W_e} w_{ns} s_{ij,i} dW = - \int_{W_e} w_{ns,i} s_{ij} dW + \int_{G_e} w_{ns} s_{ij} n_j dG \qquad (4.12)$$

Substituting (4.12) in (4.10) the elementary residual formulation reads as:

$$\int_{W_e} w_{ns}\left(r u_j u_{i,j}\right) dW + \int_{W_e} w_{ns,i} s_{ij} dW - \int_{G_e} w_{ns} s_{ij} n_i dG - \int_{W_e} w_{ns} f_i dW + \qquad (4.13)$$

$$+ \int_{W_e} w_c u_{i,i} dW = 0$$

The composition of (4.13) written for each element leads to the following global formulation:

$$\sum_e \int_{W_e} w_{ns}\left(r\,u_j\,u_{i,j}\right)dW + \sum_e \int_{W_e} w_{ns,i}\,s_{ij}\,dW - \sum_e \int_{G_{int}} w_{ns}[\,s_{ij}\,n_i\,]dG + \sum_e \int_{W_e} w_c\,u_{i,i}\,dW =$$
$$\sum_e \int_{W_e} w_{ns}\,f_i\,dW + \int_{G_h} w_h\,s_{ij}\,n_i\,dG$$

(4.14)

where the collection of elementary fluxes across the inter-element boundaries gives rise to the following integrals:

$$S_e \int_{G_e} w_{ns}\,s_{ij}\,n_i\,dG = S_e \int_{G_{int}} w_{ns}\left[s_{ij}\,n_i\right]dG + \int_{G_h} w_h\,s_{ij}\,n_i\,dG$$

(4.15)

In (4.15) is interesting to distinguish a first inter-element balance term (consequence of the element-wise continuity of basis function) and a second flux integral across the global boundary portion G_h were the natural conditions are applied. It is worth to note that approaching the inter-elementary boundaries the first order derivatives of the adopted trial and weighting functions give rise to equal absolute values with opposite sign. Such conditions show the perfect balance of diffusive inter-elementary integrals:

$$s_n^+ = s_n^- \;\rightarrow\; \left[s_{ij}\,n_j\right] = 0$$

(4.16)

As a consequence the discretized residual formulation of Navier-Stokes problem (4.14) could be written as:

$$\sum_e \int_{W_e} w_{ns}\left(r\,u_j\,u_{i,j}\right)dW + \sum_e \int_{W_e} w_{ns,i}\,s_{ij}\,dW + \sum_e \int_{W_e} w_c\,u_{i,i}\,dW = \sum_e \int_{W_e} w_{ns}\,f_i\,dW + \int_{G_h} w_h\,h_i\,dG$$

(4.17)

As a matter of fact the substitution of natural boundary conditions ($s_{ij} = h_i$ on G_h) in (4.14), has permitted to demonstrate the equivalence between the Navier-Stokes residual formulations under exam:

- *weak formulation*, defined on the global computational domain as a sort of energetic condition to find the solution;

- *residual formulation*, defined on an element basis as a local orthogonality condition between the residuals and the variations

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It is thus possible to conclude that the modeling of flow behavior using an approximate finite element method usually requires the ensuing steps:

- the approximate representation of continuous computational domain W, as the composition of small sub-domains called 'elements' W_e;

- the definition of approximate solution, obtained by the interpolation of the unknown values in a finite number of nodal points defined in each element using a collection of basis functions;

- the definition of an integral equation for each unknown variable, by use of a residual principle.

5.   Finite element interpolation functions

One of the crucial aspect in the application of finite element methods to CFD concerns with the choice of *shape* or *basis* functions, that is the interpolating functions used to approximate the solutions and their variations on the discretized computational domain. The structure of such functions and their order determine the accuracy of FEM.

The shape functions could be given by both polynomials (with several orders) and product of polynomials to trigonometric or exponential functions. For instance the adoption of polynomial expressions allows a linear approximation of a variable in a quadrangular two-dimensional element by use of its values at the four corner nodes. Again in a quadrangular two-dimensional element a polynomial could simulate a quadratic law just adding five nodes, (four nodes in the middle of each element edge and the last in the element center). Similarly work polynomial expressions more complex as the Lagrange ones, allowing different order of approximation in each Cartesian directions. As a consequence of the increasing complexity, shape functions with growing order permits a better approximation of unknowns behavior and thus leading to the use of a reduced number of elements and nodal points (that is the number of problem DOFs), preserving the accuracy of the FEM. On the other hand the use of higher-order polynomials for the collection of finite element shape functions leads to increased coding difficulties.

Such considerations state clearly that the choice of finite element and of the associated function spaces must be the consequence of a compromise between the accuracy of the FEM and its computational cost (in terms of coding complexity, CPU time and storage requirements).

As far as the geometric shape of the elements is concerned, often the discretized modeling of complex geometries introduces distortion of the sub-domains (particularly with quadrangular like shape), in order to simulate close the domain boundaries. From a computational viewpoint, it is thus mandatory to introduce a mapping operation able to simply correlate the spatial description of each element (Cartesian, spherical, etc. nodes coordinates) with a normal or logic system of coordinates. With the great advantage that in a normalized geometry representation, indicated as *(ξ, η, ζ)*, the corner nodes coordinates have always unit value (negative or positive). Fig. 5.1 shows a two-dimensional logic element, while the following Fig. 5.2 describe the correspondence Cartesian and logic coordinate systems through a mapping operator for two-dimensional geometries.
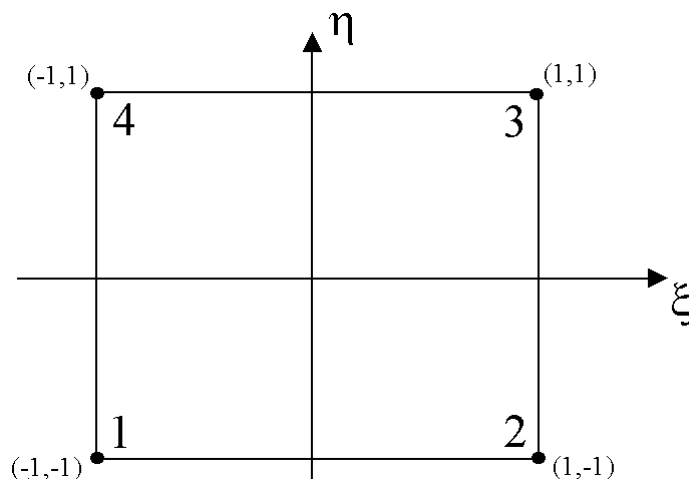
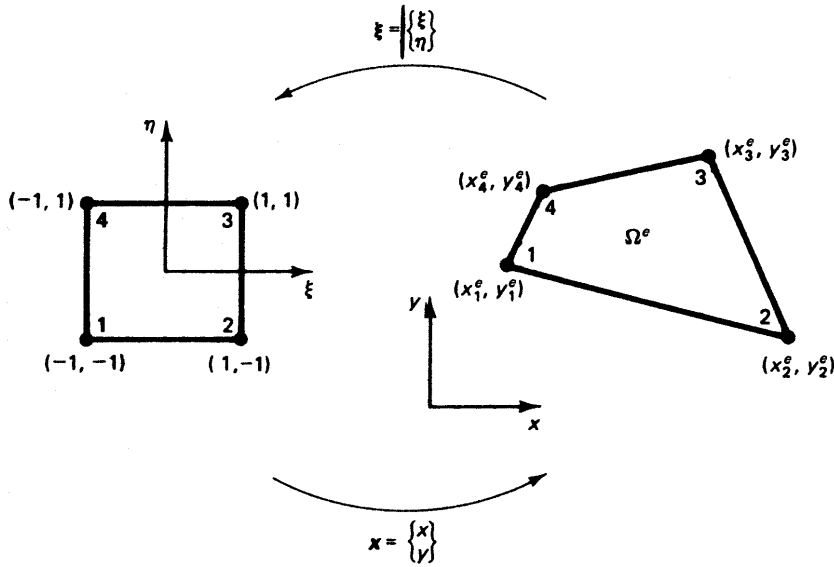**Fig. 5.1 - Two-dimensional element in logic reference**

$$\boldsymbol{\xi} = \left\{ \begin{matrix} \xi \\ \eta \end{matrix} \right\}$$

$$\boldsymbol{x} = \left\{ \begin{matrix} x \\ y \end{matrix} \right\}$$

Fig. 5.2 – Link between Cartesian (x-y) and logic (ξ–η) frame of references

The mapping operator is generally established through an analytical correspondence between the global and the local normalized systems of coordinates (Fig. 5.2) with the following general functional relationship:

$$\left\{ \begin{matrix} x \\ y \\ z \end{matrix} \right\} = f \left\{ \begin{matrix} \boldsymbol{x} \\ \boldsymbol{h} \\ \boldsymbol{V} \end{matrix} \right\} \tag{5.1}$$

Where the function f defines analytically the *coordinates transformation operator* and it introduces a bi-univocal correspondence between the elements of the discretized domain and the rectangular normalized elements.

The simplicity of the logic representation of elementary domain together with their universal applicability to element of generic shape, suggest in a FE based CFD code the use of the logical frame of reference as the basis for the definition of the approximate integral formulation (e.g. local shape function to build both trial and weighting functions collections). Element by element, the real Cartesian geometry would be then constructed using appropriate transformation operator. In detail the link between the introduced system of coordinates could be expressed as follows:

$$x(\boldsymbol{x},\boldsymbol{h},\boldsymbol{V}) = \sum_{i=1}^{nel} N_i(\boldsymbol{x},\boldsymbol{h},\boldsymbol{V})x_i \tag{5.2}$$

$$y(\boldsymbol{x},\boldsymbol{h},\boldsymbol{V}) = \sum_{i=1}^{nel} N_i(\boldsymbol{x},\boldsymbol{h},\boldsymbol{V})y_i \tag{5.3}$$

$$z(\boldsymbol{x},\boldsymbol{h},\boldsymbol{V}) = \sum_{i=1}^{nel} N_i(\boldsymbol{x},\boldsymbol{h},\boldsymbol{V})z_i \tag{5.4}$$

where *nel* is the number of element nodes, N_i are the appropriate interpolating functions, (x_i,y_i,z_i) are the nodal coordinates le coordinate in the global frame of reference.

For instance, with reference to a two-dimensional element with four nodes (Fig. 5.1) the related linear shape functions N_i could be defined as:

$$x = a_0 + a_1 x + a_2 h + a_3 xh \qquad (5.5)$$

$$y = b_0 + b_1 x + b_2 h + b_3 xh \qquad (5.6)$$

with the polynomial coefficients a and b that could be obtained applying the (5.5) and (5.6) at each corner nodes where the global-to-login coordinate transformation is known. Such a position leads to the following relations:

$$x(x_a, h_a) = \sum_{i=1}^{N} N_i(x_a, h_a)x_i = x_a \qquad (5.7)$$

$$y(x_a, h_a) = \sum_{i=1}^{N} N_i(x_a, h_a)y_i = y_a \qquad (5.8)$$

Writing the above relations for each element node a system of equations for the unknown polynomial coefficients a and b (in three-dimensions a, b and g) is finally obtained.

The shape functions could be defined using an alternative procedure called *direct formulation,* that is based on the construction of the collection of interpolating basis according to the set of general properties they have to fulfill:

- the function related to the generic element node i mist be such that $N_i(i) = 1$ and $N_i(j) = 0$ for $j \neq i$;
- the behavior of the function along the element boundaries must guarantee the continuity with neighboring elements.

Just to give an example again with reference to the element shown in Fig. 5.1, the shape function for the upper-right node (normalized coordinates *(1,1)*) could be defined as the product *(x + 1)(h + 1) /4*. As a matter of fact it assumes the unit value at the position *(1,1)* and zero values at the other nodal positions where at least one of the normalized corrdinates is set equal to *-1*. Furthermore the defined function shows a linear variation on the element edges and by that way guarantees the continuity with the neighboring domains.

The *isoparametric elements* are commonly adopted in the developed FE codes. The name 'isoparametric' derives from the fact that the same parametric function which describes the geometry may be used for interpolating spatial variations of a variable (u, v, T, p,....) within an element. In general:

$$x(x, h, V) = \sum_{i=1}^{nel} N_i(x, h, V)x_i \qquad (5.9)$$

$$u(x, h, V) = \sum_{i=1}^{nel} F_i(x, h, V)u_i \qquad (5.10)$$

and for an isoparametric element $F_i(x, h, z) = N_i(x, h, z)$.

The last problem to be solved in the contruction of FE based CFD code, is the definition of shape function derivatives in the global system of coordinates (x, y, z) necessary to the definition of element coefficient matrices. Such matrices contain the global unknowns and variations derivatives, that are of difficult direct computation while they offer a simple definition in the logic frame of reference. It is thus mandatory to find a logic-to-global transformation tool for the derivatives.

Let compute the shape function first derivatives with reference to the normalized coordinates x, h, and z it is possible to write:

$$\frac{\partial F}{\partial \mathbf{x}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \mathbf{x}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \mathbf{x}} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial \mathbf{x}} \tag{5.11}$$

$$\frac{\partial F}{\partial \mathbf{h}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \mathbf{h}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \mathbf{h}} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial \mathbf{h}} \tag{5.12}$$

$$\frac{\partial F}{\partial \mathbf{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \mathbf{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \mathbf{z}} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial \mathbf{z}} \tag{5.13}$$

or in a matrix form:

$$
\begin{Bmatrix} \dfrac{\partial F}{\partial \mathbf{x}} \\[4pt] \dfrac{\partial F}{\partial \mathbf{h}} \\[4pt] \dfrac{\partial F}{\partial \mathbf{V}} \end{Bmatrix}
=
\begin{bmatrix}
\dfrac{\partial x}{\partial \mathbf{x}} & \dfrac{\partial y}{\partial \mathbf{x}} & \dfrac{\partial z}{\partial \mathbf{x}} \\[4pt]
\dfrac{\partial x}{\partial \mathbf{h}} & \dfrac{\partial x}{\partial \mathbf{h}} & \dfrac{\partial x}{\partial \mathbf{h}} \\[4pt]
\dfrac{\partial x}{\partial \mathbf{z}} & \dfrac{\partial y}{\partial \mathbf{z}} & \dfrac{\partial z}{\partial \mathbf{z}}
\end{bmatrix}
\begin{Bmatrix} \dfrac{\partial F}{\partial x} \\[4pt] \dfrac{\partial F}{\partial y} \\[4pt] \dfrac{\partial F}{\partial z} \end{Bmatrix}
= J \begin{Bmatrix} \dfrac{\partial F}{\partial x} \\[4pt] \dfrac{\partial F}{\partial y} \\[4pt] \dfrac{\partial F}{\partial z} \end{Bmatrix}
\tag{5.14}
$$

where J is called the Jacobian and defined the searched transformation operator.

As a matter of fact the global derivatives could be defined after the computation of the inverse Jacobian J^{-1}, as:

$$
\begin{Bmatrix} \dfrac{\partial F}{\partial x} \\[4pt] \dfrac{\partial F}{\partial y} \\[4pt] \dfrac{\partial F}{\partial z} \end{Bmatrix}
= J^{-1}
\begin{Bmatrix} \dfrac{\partial F}{\partial \mathbf{x}} \\[4pt] \dfrac{\partial F}{\partial \mathbf{h}} \\[4pt] \dfrac{\partial F}{\partial \mathbf{V}} \end{Bmatrix}
\tag{5.15}
$$

From an algorithmic point of view, the demonstrated simplicity of treating the interpolating functions and their derivatives in the logic reference leads to the computation of the integral terms that define the coefficient matrix (on an element level) in a normalized control domain. Such a technique requires of course a coordinate transformation able to transfer the differential area or volume from the global to the logic reference. Of course the same transformation must involve the extremes of integration. Thus defined K as a generic term of the element matrix the transformation of integral could be written as:

$$\int_V K\,dV = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K[\det J]\,d\mathbf{x}\,d\mathbf{h}\,d\mathbf{V} \tag{5.16}$$

5.1 Interpolation spaces in XENIOS

A sample of the shape function implemented in the CFD code XENIOS is here given. The shape function are defined with their first order derivatives for two-dimensional element with both bi-linear or bi-quadratic approximation. Such functions have been defined by use of the direct formulation already discussed.

The four nodes quadrangular element is shown in Fig. 5.3, together with the node numbering.

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Fig. 5.3 - Quadrangular linear element, local node numbering

The related bi-linear shape functions are:

$$
\begin{bmatrix}
\dfrac{1}{4}(1-\mathbf{x})(1-\mathbf{h}) \\
\dfrac{1}{4}(1+\mathbf{x})(1-\mathbf{h}) \\
\dfrac{1}{4}(1+\mathbf{x})(1+\mathbf{h}) \\
\dfrac{1}{4}(1-\mathbf{x})(1+\mathbf{h})
\end{bmatrix}
\tag{5.17}
$$

The first order derivatives of the bi-linear shape functions (5.17) are:

$$
\frac{\partial \mathbf{f}}{\partial \mathbf{x}} =
\begin{bmatrix}
-\dfrac{1}{4}(1-\mathbf{h}) \\
+\dfrac{1}{4}(1-\mathbf{h}) \\
+\dfrac{1}{4}(1+\mathbf{h}) \\
-\dfrac{1}{4}(1+\mathbf{h})
\end{bmatrix}
\tag{5.18.a}
$$

$$
\frac{\partial \mathbf{f}}{\partial \mathbf{h}} =
\begin{bmatrix}
-\dfrac{1}{4}(1-\mathbf{x}) \\
-\dfrac{1}{4}(1+\mathbf{x}) \\
+\dfrac{1}{4}(1+\mathbf{x}) \\
+\dfrac{1}{4}(1-\mathbf{x})
\end{bmatrix}
\tag{5.18.b}
$$

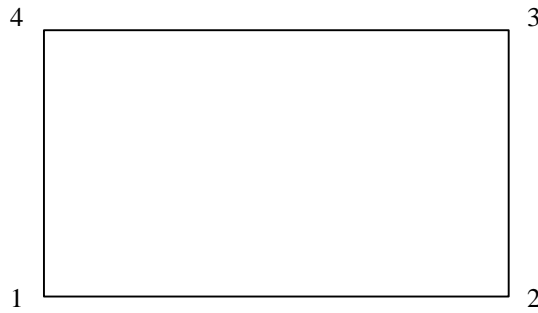The bi-quadratic nine nodes element has the local numbering shown in Fig. 5.4.
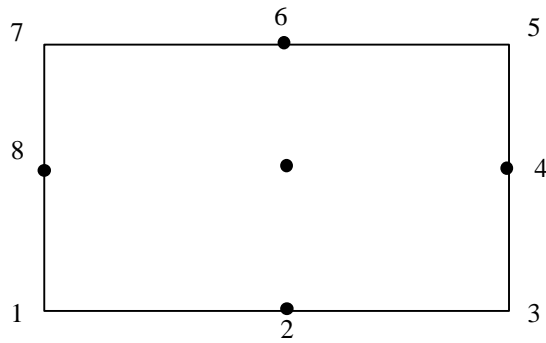
Fig. 5.4 - Quadrangular linear element, local node numbering

The related bi-quadratic shape functions are:

$$
\begin{bmatrix}
\dfrac{1}{4}\boldsymbol{xh}(1-\boldsymbol{x})(1-\boldsymbol{h}) \\[2mm]
-\dfrac{1}{2}\boldsymbol{h}\left(1-\boldsymbol{x}^2\right)(1-\boldsymbol{h}) \\[2mm]
-\dfrac{1}{4}\boldsymbol{xh}(1+\boldsymbol{x})(1-\boldsymbol{h}) \\[2mm]
\dfrac{1}{2}\boldsymbol{x}\left(1+\boldsymbol{x}\ \right)\left(1-\boldsymbol{h}^2\right) \\[2mm]
\dfrac{1}{4}\boldsymbol{xh}(1+\boldsymbol{x})(1+\boldsymbol{h}) \\[2mm]
\dfrac{1}{2}\boldsymbol{h}\left(1-\boldsymbol{x}^2\right)(1+\boldsymbol{h}) \\[2mm]
-\dfrac{1}{4}\boldsymbol{xh}(1-\boldsymbol{x})(1+\boldsymbol{h}) \\[2mm]
-\dfrac{1}{2}\boldsymbol{x}\left(1-\boldsymbol{x}\ \right)\left(1-\boldsymbol{h}^2\right) \\[2mm]
\left(1-\boldsymbol{x}^2\right)\left(1-\boldsymbol{h}^2\right)
\end{bmatrix}
\tag{5.19}
$$

with the following first order derivatives:

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{1}{4}\boldsymbol{h}(1-2\boldsymbol{x})(1-\boldsymbol{h}) \\ \boldsymbol{hx}(1-\boldsymbol{h}) \\ -\frac{1}{4}\boldsymbol{h}(1+2\boldsymbol{x})(1-\boldsymbol{h}) \\ \frac{1}{2}(1+2\boldsymbol{x})(1-\boldsymbol{h}^2) \\ \frac{1}{4}\boldsymbol{h}(1+2\boldsymbol{x})(1+\boldsymbol{h}) \\ -\boldsymbol{hx}(1+\boldsymbol{h}) \\ -\frac{1}{4}\boldsymbol{h}(1-2\boldsymbol{x})(1+\boldsymbol{h}) \\ -\frac{1}{2}(1-2\boldsymbol{x})(1-\boldsymbol{h}^2) \\ -2\boldsymbol{x}(1-\boldsymbol{h}^2) \end{bmatrix} \qquad (5.20.\text{a})$$

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{h}} = \begin{bmatrix} \frac{1}{4}\boldsymbol{x}(1-\boldsymbol{x})(1-2\boldsymbol{h}) \\ -\frac{1}{2}(1-\boldsymbol{x}^2)(1-2\boldsymbol{h}) \\ -\frac{1}{4}\boldsymbol{x}(1+\boldsymbol{x})(1-2\boldsymbol{h}) \\ -\boldsymbol{xh}(1+\boldsymbol{x}) \\ \frac{1}{4}\boldsymbol{x}(1+\boldsymbol{x})(1+2\boldsymbol{h}) \\ \frac{1}{2}(1-\boldsymbol{x}^2)(1+2\boldsymbol{h}) \\ -\frac{1}{4}\boldsymbol{x}(1-\boldsymbol{x})(1+2\boldsymbol{h}) \\ \boldsymbol{xh}(1-\boldsymbol{x}) \\ -2\boldsymbol{h}(1-\boldsymbol{x}^2) \end{bmatrix} \qquad (5.20.\text{b})$$

6. Stabilized finite element formulation for advective-diffusive flows

The stabilization methods, that will be discussed in following Chapters, define numerical tools able to correct the instability origins that affect originally the finite element Galerkin fromulation of incompressible Navier-Stokes problem. Their presentation is here carried out with reference to two different extreme flow conditions:

- the purely *advective flow limit*, characterized by the localization of advective transport of variable along the streamlines;

- the purely *diffusive flow limit* (*Stokes flow*), characterized by pressure instability related to the incompressibility constraint.

6.1 *The stabilization of convection dominated flow*

The solution of fluid dynamic problems using symmetric Galerkin residual formulation loses the best approximation properties showed in structural and thermal problems (Hughes et al., 1982). As a consequence of the convective transport of fluid variables through the computaional domain the coefficient matrix associated with the governing equations contains first order non symmetric terms.

The difficulties that arise in simulating velocity fields, strongly asymmetric, using symmetric operators such that the elementary Galerkin shape functions (or equivalently centred finite differences stencil) give rise to the presence of spurious oscillations of velocity on contiguous node (called 'wiggles').

A simple way to comprehend such a numerical behavior could be searched on the basis of the demonstrated equivalence between linear Galerkin finite element approximation and the centered finite difference (Hughes et al, 1982) (Leonard, 1979).

The modeling with centered finite difference (that is symmetric operators) of first order derivatives does not lead to an intrinsic stability.

Let consider, for instance, a convective one-dimensional problem where the velocity u transport a scalar j through the domain $W \subset R^1$:

$$u \cdot \frac{\partial j}{\partial x} \tag{6.1}$$

The stability condition generally requires that each wrong variations of the value of the transported scalar j should results in a variations of the convective governing term able to correc and compensate such error. That is the stability condition could be expressed as:

$$\frac{\partial(convective.term)}{\partial j} < 0 \tag{6.2}$$

Let now consider the form that (6.1) assumes if a centered finite difference approximation is used. Indicating with the sub-scripts (*i-1*) and (*i+1*) the grid points located respectively upwind and downwind from the point (*i*) where the term has to be computed, it is possible to write:

$$u \cdot \frac{\partial j}{\partial x} \approx u \frac{j_{i+1} - j_{i-1}}{2 Dx} \tag{6.3}$$

that clearly shows that the sensitivity of convective term to the variations of scalar j_i is zero.

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Thus, it could be concluded that the modeling of convective term, appearing in the momentum equations as well as in the conservation equations of turbulent variables, carried out with operators spatially symmetric introduce in the algorithm a neutral stability. That is the modeled convective term is unable to feel the direction of propagation along the grid of the convective signals.

The first idea developed for recovering the algorithm stability proceeds modeling the convective term by use of upwind finite differences scheme, so that the spatial derivatives of the variable in each nodal position (*i*) explictly depends from the value assumed at the node itself (*i*) and at the node with an upwind location. In such a way that a directional stability is recovered at expense of accuracy of the formulation.

Let now demonstrate that an equivalent stabilization could be carried out by the composition of a convective term modeled with centered finite difference stencil and an artificial diffusive like term. With reference to a one-dimensional problem it could be written:

$$u \cdot \frac{\partial \varphi}{\partial x} \approx u \frac{\varphi_i - \varphi_{i-1}}{Dx} = u \frac{\varphi_{i+1} - \varphi_{i-1}}{2Dx} + \frac{uDx}{2} \frac{-\varphi_{i+1} + 2\varphi_i - \varphi_{i-1}}{Dx^2} \qquad (6.4)$$

where

$$\tilde{k} = \frac{uDx}{2}$$

defines a numerical diffusivity that directly depends on the magnitude of convective phenomena relative to the characteristic grid dimension (*Dx*).

The interpretation of the upwind differencing technique using the artificial diffusivity approach, represents the link that originally gives rise to the possibility of implementation of such stabilization methods in the framework of a residual finite element formulations. It is therefore with the goal of correcting the typical under-diffusivity of Galerkin scheme that the stabilization technique described in the following Chapter have been developed.

Non-consistent stabilization methods, Upwind and Streamline Upwind

The stabilization of Galerkin finite element formulation is based on the introductionof an artificial balancing integral able of correcting the negative diffusivity of the residual method. An appropriate choice of the artificial contribution intensity could lead, in a one-dimensional case, to the simulation of exact numerical solution. Such upwind schemes are called *optimal* (Brooks and Hughes, 1982).

The artificial diffusivity is in the *optimal upwind scalar scheme* defined using the following expressions:

$$\tilde{k} = \left(\frac{|u|h}{2}\right)z$$

$$z = coth(a) - 1/a$$

$$a = \frac{|u|h}{2k}$$

(6.5)

where, for a meaningful extension to multi-dimensional case:

$|u|$, is the absolute value of the local velocity;

h, is the characteristich element dimensio;

a, is the elementary or grid Peclet number;

k, is the physical fluid diffusivity;

z, is a stabilization coefficient able to modify the intensity of artificial diffusivity with reference to the magnitude of convective transport phenomena.

The application of *upwind scalar schemes* to multi-dimensional flow conditions leads often to non controlled over-diffused solution in particular showing non physical cross wind diffusion, because of the isotropic character of the balancing operator.

In order to eliminate the drawbacks shown by the scalar schemes has been then developed the *streamline upwind* technique where the upwind effect is concentrated in principle along the streamline direction. In such a method the balancing operator, again with the form of a diffusive term, acts exclusively in the streamline direction as an anisotropic artificial diffusivity. The artificial diffusivity assumes therefore a tensorial character and could be expressed as follows:

$$\tilde{k}_{ij} = \tilde{k} \cdot \overline{u}_i \overline{u}_j$$

(6.6)

where:

$\overline{u}_j = u_j / \|u\|$, defines the velocity components unit vector,

$\|u\|^2 = u_i u_i$, is the velocty norm,

$\tilde{k}$, is the artificial dissusivity already defined with reference to the scalar upwing techniques.

It is now interesting the analysis of the form of the tensorial balancing term. Let concentrate the analysis to steady and incompressible Navier-Stokes equations.

Consider the divergence of stress tensor appearing in the molecular diffusive term, its symmetric part could be written as:

$$(2k_{ij}(u_{i,j} + u_{j,i})),_j = 2(k_{ij}(u_{i,j})),_j + 2(k_{ij}(u_{j,i})),_j$$

(6.7)

By simply reverse the derivation order of the second term and imposing the incompressibility condition, divergence free velocity field, the (6.7) could be modified as follow:

$$2(k_{ij}(\boldsymbol{u_{i,j}})),_{j} +2(k_{ij}(\boldsymbol{u_{j,i}})),_{j} = 2(k_{ij}(\boldsymbol{u_{i,j}})),_{j} +2(k_{ij}(\boldsymbol{u_{j,j}})),_{i}$$
$$2(k_{ij}(\boldsymbol{u_{i,j}})),_{j} +2(k_{ij}(\boldsymbol{u_{j,i}})),_{j} = 2(k_{ij}(\boldsymbol{u_{i,j}})),_{j}$$

(6.8)

with

$$\boldsymbol{u}_{j,j} = 0$$

On the basis of the development, the diffusive integral term in the Galerkin residual formulationof Navier Stokes problem (4.14), (4.17) could now be written as:

$$\int_{W} \boldsymbol{w}_{ns,i}\, \boldsymbol{s}_{ij}\, dW = \int_{W} \boldsymbol{w}_{ns,i} \left(k_{ij} + \widetilde{k}_{ij} \right) \boldsymbol{u}_{i,j}\, dW$$

where explictly appears a diffusivity obtained as the sum of physical and artificial contribution:

$$\int_{W} \boldsymbol{w}_{ns,i} k_{ij}\, \boldsymbol{s}_{ij}\, dW + \int_{W} \boldsymbol{w}_{ns,i}\, \widetilde{k}_{ij}\, \boldsymbol{u}_{i,j}\, dW$$

(6.9)

Introducing in (6.9) the artificial tensorial diffusivity expression the stabilization integral becomes:

$$\int_{W} \boldsymbol{w}_{ns,i}\, \widetilde{k}\, \overline{\boldsymbol{u}}_{i} \overline{\boldsymbol{u}}_{j}\, \boldsymbol{u}_{i,j}\, dW$$

(6.10)

and substituting the versor $\overline{\boldsymbol{u}}_{j}$ with its definition

$$\overline{\boldsymbol{u}}_{j} = \boldsymbol{u}_{j} \big/ \|\boldsymbol{u}\|$$

it is possible to write:

$$\int_{W} \boldsymbol{w}_{ns,i}\, \frac{\overline{\boldsymbol{u}}_{i}}{\|\boldsymbol{u}\|}\, \widetilde{k}\, \boldsymbol{u}_{j}\, \boldsymbol{u}_{i,j}\, dW$$

(6.11)

What clearly appears from the (6.11) is that the tensorial stabilization term has the form of the convective integral that must be controlled.

The analysis carried out has demonstrated the equivalence between the two different approach developed to obtain a *streamline upwind* stabilization in the ambit of a finite element residual formulation: from one hand the classical approach that proceeds adding a diffusive balancing integral, from the other hand the intervention on the convective integral in such a way that the original Galerkin weight is modified by a perturbation depending from the stabilization parameters.

Residual stabilized formulation *Streamline Upwind - Petrov Galerkin (SU/PG)*

The consistency of the stabilization methods could be recovered extending the weights perturbation, limited to the convective integral in the *streamline upwind* scheme, to each term that is contained in the residual Navier-Stokes problem formulation (4.17). In such a way the build residual structure assumes the character of a Petrov-Galerkin formulation, due to the introduction of different function spaces used for the approximation of the solution and of the variations. Such a residual stabilized finite element formulation for convection dominated flows is called *streamline upwind - Petrov Galerkin* (*SU/PG*) (Brooks and Hughes, 1982).

The established consistency of integral governing equations move definitively the *streamline upwind - Petrov Galerkin* from the classical upwind methods closely linked to the finite difference scalar upwind concept. In such a way that the SUPG formulation is not subject to the artificial diffusion criticism associated to the early stabilization technique.

Let consider a flow region $W \in R^{nsd}$ (*nsd* is the number of space dimension), which has a boundary G defined by piecewise continuous functions. Consider also a point x_i ($i = 1, \ldots, nsd$) belonging $\overline{W}$, and define n_i as the component along i direction of the unit vector normal to G (positive orientation toward the inner of the domain). The domain boundary G is shared in two subset G_g and G_h, that satisfy the following relation:

$$\overline{G_g \cup G_h} = G \tag{6.12}$$

$$G_g \cap G_h = \varnothing \tag{6.13}$$

As already mentioned, the finite element method proceeds by subdividing the domain W into a finite number of elements *numel* W_e, dove $e = 1,2 \ldots, numel$. Let now define G_e as the boundary fo the element W_e, the discretization follows the ensuing properties:

$$\cup_e \overline{W}_e = W \tag{6.14}$$

$$\cap_e \overline{W}_e = \varnothing \tag{6.15}$$

with an interior boundary such as:

$$G_{int} = \cup_e G_e - G \tag{6.16}$$

Recall also the PDE that governs a steady and incompressible Navier-Stokes boundary problem:

$$r u_j u_{i,j} = s_{ij,j} + f_i \tag{6.17}$$

with the following set of constraints:

$u_{i,i} = 0$, the incompressibility constraint; $\qquad\qquad$ (6.18)

$u_i = g_i \to G_g$, essential Dirichlet boundary conditions;

$s_{ij} n_i = h_i \to G_h$, natural Neumann boundary conditions.

The classical Galerkin residual method adopts identical collections of trial and weighting functions, so that the weights are then continuous across the inter-element boundaries. As a matter of fact such property is lost

when a Petrov-Galerkin (SUPG) formulation is used due to the perturbation of the set of weights that modify the original Galerkin functions on an element basis as follows:

$$w_{ns}{'} = w_{ns} + p_{ns} \qquad (6.19)$$

where w_{ns} is the Galerkin weight applied to the momentum equations and p_{ns} is the stabilizing *streamline upwind* like contribution. The application of such a perturbation on contiguous elements introduces the discontinuity of weighting functions already mentioned. However the perturbation function still fulfills the integrability property on an element scale.

Let consider a point x belonging to the interior boundary G_{int} and, arbitrarily, establish a positive orientation for the normal direction across the boundary. Define also n^+ and n^- as the unit vector normal to G_{int} in the considered nodal position x, it could be written that:

$$n^- = - n^+$$

Introducing for simplicity a term that accounts for the sum of convective and diffusive fluxes:

$$c_i = c_i^a + c_i^d = r u_j u_{i,j} - s_{ij} \qquad (6.20)$$

Is possible to show that the jump of χ_i at the considered nodal point boundary x across the neighboring elements, defined as:

$$[c_n] = \left(c_i^+ - c_i^-\right)n_i^+ = c_i^+ \, n_i^+ + c_i^- \, n_i^- \qquad (6.21)$$

is an invariant with respect to the adopted sign convention for G_{int}.

On the basis of such an introduction, the application of perturbed weighting function on the basis of the SUPG method to the Navier-Stokes problem leads to the following residual stabilized formulation:

$$\sum_e \int_{W_e} [\, w_{ns}(r u_j u_{i,j}) + w_{ns,i} s_{ij} - w_{ns} f_i \,]dW + \sum_e \int_{W_e} [\, p_{ns}(r u_j u_{i,j}) - p_{ns} s_{ij,j} - p_{ns} f_i \,]dW$$
$$- \sum_e \int_{G_{int}} w_{ns}[\, s_{ij} n_i \,]dG - \int_{G_h} w_h h_i \, dG = 0 \qquad (6.22)$$

or in an equivalent way:

$$\sum_e \int_{W_e} w'_{ns}[\,(r u_j u_{i,j}) - s_{ij,j} - f_i\,]dW - \sum_e \int_{G_{int}} w_{ns}[\, s_{ij} n_i \,]dG - \int_{G_h} w_h(s_{ij} n_i - h_i)dG = 0 \qquad (6.23)$$

It is worth to note that from the obtained residual form of the integral problem is again possible to extract the original differential expression of the Navier-Stokes boundary problem. Imposing the annihilation of the whole integrand functions, the Euler-Lagrange conditions of the residual formulation could be extracted giving the following system of equations:

$$r u_j u_{i,j} + s_{ij,j} - f_i = 0$$
$$(s_{ij} n_i - h_i) = 0 \cdot \rightarrow \cdot G_h \qquad (6.24)$$
$$[s_{ij} n_i] = 0 \cdot \rightarrow \cdot G_{int}$$

The differential formulation obtained (6.24) is simply modified with respect to the original PDE form by a continuity condition of diffusive fluxes across the inter-element boundary.

The consistency of the *SU/PG* stabilization method remains thus demonstrated.

The fundamental aspect that characterized the obtained stabilized finite element formulation in comparison to the classical upwind schemes is that the *streamline upwind* perturbation function p_{ns} plays is role limited to the element interior where the perturbation itself is continuous. The function p_{ns} does not alter the continuity condition of diffusive flux across the element boundaries as well as the natural boundary condition h, while it acts implicitly on the effect of the Dirichlet essential conditions through the convective term (Brooks and Hughes, 1982).

It is furthermore interesting to note that, in case of adoption of linear shape functions and the domain discretization leads to rectangular elements the divergence of the diffusive fluxes is equal to zero:

$$s_{ij,j} = 0$$

that is the function p_{ns} does not play any stabilization effect on the diffusive integral.

The expression of *streamline upwind* perturbation p_{ns} of weights is then defined starting from the modification of convective weight obtained discussing the non consistent *streamline upwind* method (6.11). The shown equivalence between the modification of the convective weight and the introduction of a tensorial diffusive balancing integral permits to write the following:

$$p_{ns} = \tilde{k} \, \overline{u}_j \, w_{ns,j} \, / \, \|u\| \tag{6.25}$$

Fig. 6.1 shows the effect of the perturbation of a linear Galerkin function.
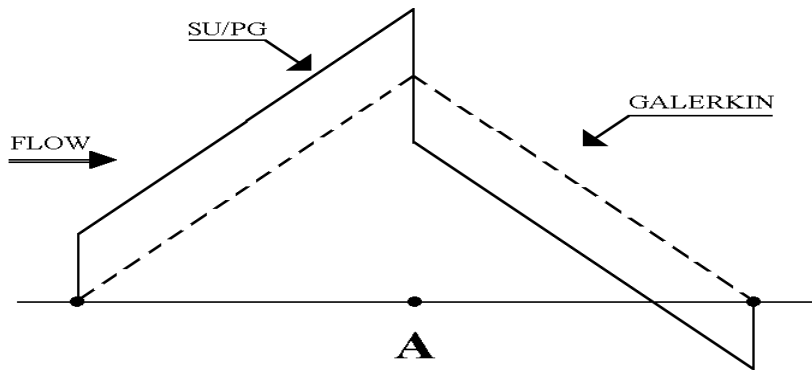
Fig. 6.1. Comparison between linear Galerkin and Petrov-Galerkin weights

The coefficient $\tilde{k}$ again is an artificial diffusivity that could be defined in several ways. Still remains valid, as verified by numerical computations, that its absolute value as little importance if compared to the structure of the perturbation function p_{ns} defined in (6.25). However, is here reported the expression for $\tilde{k}$ implemented in XENIOS for multi-dimensional computations:

$$\tilde{k} = \frac{\left(\overline{x} \, u_x \, h_x + \overline{h} \, u_h \, h_h + V \, u_V \, h_V \right)}{2} \tag{6.26}$$

with:

$$\boldsymbol{x} = coth(\boldsymbol{a}_{\boldsymbol{x}}) - 1/\boldsymbol{a}_{\boldsymbol{x}}$$

$$\boldsymbol{h} = coth(\boldsymbol{a}_{\boldsymbol{h}}) - 1/\boldsymbol{a}_{\boldsymbol{h}}$$

$$\boldsymbol{V} = coth(\boldsymbol{a}_{\boldsymbol{V}}) - 1/\boldsymbol{a}_{\boldsymbol{V}}$$

and:

$$\boldsymbol{a}_{\boldsymbol{x}} = \frac{|u_{\boldsymbol{x}}|h_{\boldsymbol{x}}}{2k}$$

$$\boldsymbol{a}_{\boldsymbol{h}} = \frac{|u_{\boldsymbol{h}}|h_{\boldsymbol{h}}}{2k}$$

$$\boldsymbol{a}_{\boldsymbol{V}} = \frac{|u_{\boldsymbol{V}}|h_{\boldsymbol{V}}}{2k}$$

and:

$$u_{\boldsymbol{x}} = \boldsymbol{e}_{\boldsymbol{x}} \times \boldsymbol{u}, \; u_{\boldsymbol{h}} = \boldsymbol{e}_{\boldsymbol{h}} \times \boldsymbol{u}, \; u_{\boldsymbol{z}} = \boldsymbol{e}_{\boldsymbol{z}} \times \boldsymbol{u}.$$

In detail $\boldsymbol{e}_{\boldsymbol{x}}$, $\boldsymbol{e}_{\boldsymbol{h}}$ and $\boldsymbol{e}_{\boldsymbol{z}}$ are the unit vectors of logic elementary directions $\boldsymbol{x}$, $\boldsymbol{h}$ and $\boldsymbol{z}$ while $h_{\boldsymbol{x}}$ $h_{\boldsymbol{h}}$ hz are appropriate characteristic elementary length scales. Clearly if the weight w_{ns} is continuous across the element boundaries, the perturbation p_{ns} and as a consequence the modified weight w_{ns}' will be discontinuous.

The *Streamline Upwind - Petrov Galerkin* formulation implemented in XENIOS

The implementation of a SUPG stabilized formulation in the finite element code XENIOS has required two fundamental changes with respect to the formulation originally proposed by Brooks and Hughes (1982).

The use in XENIOS of mixed stable elements, with a quadratic interpolation for the velocity and turbulent quantities and a linear one for the pressure, gives rise to an original interpretation of the consistent stabilization scheme.

As a matter of fact the use of second order basis function leads to the presence in the advective-diffusive equations of the flow model of stabilizing integral with non-zero second order derivatives. Such as:

$$\int_{W_e} p_{ns} \boldsymbol{s}_{ij,j} dW = \int_{W_e} \frac{\tilde{k}}{\|\boldsymbol{u}\|} \bar{u}_j \, w_{ns}{}_{,j} \, \boldsymbol{s}_{ij,j} dW \qquad (6.27)$$

With a direct reference to the Navier Stokes formulation, two terms of the stabilized integral formulation have been changed:

$$\int_{W_e} p_{ns} \boldsymbol{s}_{ij,j} dW = -\int_{W_e} p_{ns} \; p_{,j} dW + \int_{W_e} p_{ns} \; (2k_{ij}(u_{i,j} + u_{j,i}))_{,j} dW \qquad (6.28)$$

The first change involve the integral term of the momentum equation proportional to the pressure gradient. The structure of the SUPG stabilizing integral in an elementary domain is:

$$\int_{W_e} \tilde{k} \frac{\overline{u}_j}{\|u\|} w_{ns,j} \; p_{,j} dW \tag{6.29}$$

The stabilizing effect is therefore a consequence of the product in each element of the spatial gradient of quadratic and linear functions. Let consider, for instance, the behavior of spatial first order derivatives of the functions w_{ns} (parabolic) and p (linear) evaluated for the central node of a one-dimensional quadratic element Fig. 6.2.
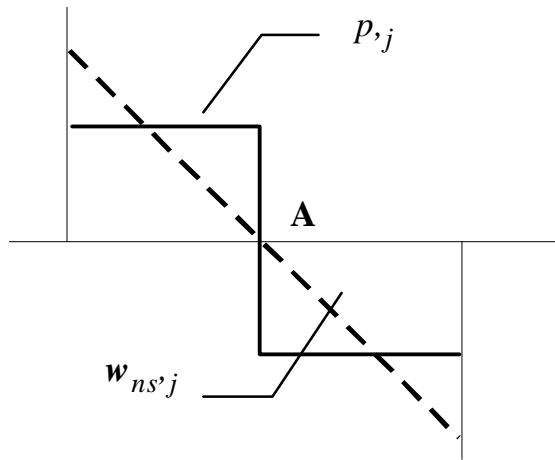

Fig. 6.2. First order derivatives of quadratic and linear shape functions

From the comparison, here limited to one dimension, becomes evident that the stabilizing contribution due to the mixed spaces of interpolation assumes an elementary value that is always positive both downwind and upwind of the node A. Such a circumstance is that the term loses its ability of introducing a streamline upwind perturbation. Its presence seems to introduce exclusively an artificial disturb that in the developed formulation is eliminated.

The second change has involved the diffusive term which implicitly influences the set of natural boundary conditions and their fulfillment.

The local elementary continuity of the introduced perturbation functions permits the application of the Green-Gauss rule on an element scale, such that a boundary integral origins that corresponds to an artificial diffusive flux of the form:

$$\int_{W_e} p_{ns} s_{ij,j} dW = -\int_{W_e} p_{ns,j} s_{ij} dW + \int_{G_e} p_{ns} \; s_{ij} n_j dW \tag{6.30}$$

Such a term is perfectly balanced along the inter-element boundaries in the interior of the computational domain, as a matter of fact each elementary artificial flux is annihilated by fluxes equal and opposite from the neighboring elements. On the contrary it should be set equal to zero on the physical boundary of the domain where it defines unphysical flux conditions.

The implemented stabilized formulation therefore adopts a technique able to enforce the set of problem natural conditions (Hansbo, 1995). Such technique imposes the annihilation of the perturbation integral in the vicinity of the domain boundary and could be formulated as:

$$\int_{W_e} d_b p_{ns} s_{ij,j} dW \tag{6.31}$$

where d_b is an switch factor build as an array with a dimension equal to the number of nodes that store unit values if the nodes belong to the interior of the domain or zero values if the nodes belong to the domain boundary:

$$d_b(i) = 1 \rightarrow \forall\ i \in G$$
$$d_b(i) = 0 \rightarrow \forall\ i \notin G$$

6.2 *The stabilization of diffusion dominated flow*

The methods that attempt to control the instability origin related to the flow incompressibility circumventing the Babuska-Brezzi condition, have as a common strategy the relaxing of free-divergence constraint on the velocity field. In such a way that the divergence of the velocity field is set equal to a small positive term, made proportional for instance to the pressure Laplacian (Brezzi and Pitkaranta, 1984):

$$u_{i,i} \propto (p_{,i})_{,i} \tag{6.32}$$

Such a technique leads of course to a lose of consistency of the stabilized residual formulation with the original PDE boundary problem.

With reference to the Stokes flow problem, that defines the diffusive limit of the more general Navier-Stokes formulation, it is possible to analyze residual formulations able to introduce a relaxation of the incompressibility constraint preserving the consistency.

Let recall the differential form of the Stokes problem for incompressible flows in a domain $W \subset R^{nsd}$, where G is the domain boundary defined by piecewise continuous functions shared in two subsets G_g and G_h, such that:

$$\overline{G_g \cup G_h} = G \tag{6.33}$$
$$G_g \cap G_h = \varnothing \tag{6.34}$$

The incompressible Stokes problem could be formulated as:

$$s_{ij,j} + f_i = 0$$
$$u_{i,i} = 0 \tag{6.35}$$

with the following boundary conditions:

$$u_i = g_i \rightarrow G_g$$
$$s_{ij} n_i = h_i \rightarrow G_h$$

The relative residual Galerkin formulation on the discretized domain is:

$$\sum_e \int_{W_e} [\,w_{s,i}\,\boldsymbol{s}_{ij} + w_s\,\boldsymbol{f}_i\,]dW - \sum_e \int_{G_{int}} w_s[\,\boldsymbol{s}_{ij}\,\boldsymbol{n}_i\,]dG - \int_{G_h} w_h\,\boldsymbol{h}_i\,dG + \sum_e \int_{W_e} w_c\,\boldsymbol{u}_{i,i}\,dW = 0 \qquad (6.36)$$

where w_s defines the Galerkin weight applied to the Stokes equations while the w_c defines the weight applied to the continuity equation.

The introduction of a perturbation of the weight w_s, which is continuous into each elementary domain, leads to the definition of a consistent stabilized residual that gives rise to a relaxation of the incompressibility constraint. Let introduce a weight function with the following structure:

$$w'_s = w_s + \boldsymbol{a}_e h_e^2 w_{c,i} \qquad (6.37)$$

where $\boldsymbol{a}_e$ defines a non-dimensional positive stability parameter and h_e is the element length scale (Hughes et al., 1986).

The presence of such perturbed weighting functions transforms the original Galerkin residual formulation into a more general Petrov - Galerkin structure:

$$\sum_e \int_{W_e} [\,w_{s,i}\,\boldsymbol{s}_{ij} + w_s\,\boldsymbol{f}_i\,]dW + \sum_e \int_{W_e} \boldsymbol{a}_e h_e^2\,w_{c,i}[\,\boldsymbol{s}_{ij,i} + \boldsymbol{f}_i\,]dW +$$
$$-\sum_e \int_{G_{int}} w_s[\,\boldsymbol{s}_{ij}\,\boldsymbol{n}_i\,]dG - \int_{G_h} w_h\,\boldsymbol{h}_i\,dG + \sum_e \int_{W_e} w_c\,\boldsymbol{u}_{i,i}\,dW = 0 \qquad (6.38)$$

Let now consider the structure of Cauchy tensor $\boldsymbol{s}_{ij}$:

$$\boldsymbol{s}_{ij} = -p\boldsymbol{d}_{ij} + 2k_{ij}(\boldsymbol{u}_{i,j} + \boldsymbol{u}_{j,i}) \qquad (6.39)$$

The stabilizing term is linked to the spatial gradient of the pressure and assumes the following expression:

$$\int_{W_e} \boldsymbol{a}_e h_e^2 w_{c,i}\, p_{,i}\, dW \qquad (6.40)$$

Transforming the (6.40) by use of an integration by parts the integral becomes:

$$\int_{W_e} \boldsymbol{a}_e h_e^2 w_{c,i}\, p_{,i}\, dW = \int_{W_e} \boldsymbol{a}_e h_e^2 w_c (\,p_{,i}\,)_{,i}\, dW - \int_{G_e} \boldsymbol{a}_e h_e^2 w_c\, p_{,i}\, n_i\, dG \qquad (6.41)$$

such expression de facto gives rise to the presence of an integral term proportional to the pressure Laplacian that due to the applied weight could be interpreted as a stabilization integral acting on the residual continuity equation. The above developments show clearly the role of the stabilization in altering the incompressibility constraint introducing a non-zero velocity field divergence.

Residual stabilized formulation *Pressure Stabilized - Petrov Galerkin (PS/PG)*

The Petrov-Galerkin method for the consistent stabilization of incompressible Stokes flow, could be exported in the more general framework of the Navier-Stokes flows as a technique to perturb the weight function applied

to the continuity equation. Such a stabilization method is called *Pressure Stabilized - Petrov Galerkin PS/PG* (Tezduyar, 1992).

The *PS/PG* term applied to the Galerkin residual continuity equation is defined as (Hughes et al., 1986) (Tezduyar, 1992):

$$\sum_e \int_{W_e} \frac{1}{r} t_{pspg} w_{c,i} [\, r u_j u_{i,j} - s_{ij,j} - f_i \,] dW \tag{6.42}$$

where t_{pspg} is a stability factor, in which all the relation with problem variable are cumulated. Such a factor in the code XENIOS assumes the following expression:

$$t_{pspg} = \frac{h}{2U} g(Re^U)$$

where:

h,   is a global length scale depending from the domain discretization;

U,   is the global scaling velocity;

Re^U,   is a Reynolds number referred to h and U;

$g(Re^U) = coth(Re^U)-1/Re^U$, is the law of dependence between the stabilization intensity and the flow regime.

It is worth to note that in the discussed PSPG formulation the residuals of the momentum equations is used to build the stabilization integral. That is the relaxation of the incompressibility constraint is made proportional to the error affecting the velocity field solution.

❧

The stabilization method here discussed has been originally developed to circumvent the Babuska - Brezzi stability condition and the need to use mixed finite element velocity-pressure spaces, thus allowing the use of equal-order interpolation spaces. Its implementation is recommendable also in presence stable pair of interpolation spaces (such the quadratic-velocity/linear-pressure) for the ensuing reasons:

- in presence of stable mixed spaces the PSPG stabilized formulation preserves the convergence properties of the original Galerkin one (Hughes et al., 1986);
- the PSPG stabilization allows the elimination of zero diagonal entries in the global coefficient matrix originally caused by the incompressibility constraint;
- the PSPG stabilization permits the achievement of faster convergence history in case of adoption of iterative solver (Hughes et al., 1986).

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