10. Euler's equation (differential momentum equation)

Inviscid flow: $\boxed{\mu = 0}$

Resultant of forces = mass · acceleration
Inviscid flow: forces caused by the pressure and field of force.

In x direction:

$$\rho\,dx\,dy\,dz\frac{dv_x}{dx} = \rho\,dx\,dy\,dz\,g_x + p\,dy\,dz - (p + \frac{\partial p}{\partial x}dx)\,dy\,dz$$

$$\frac{dv_x}{dx} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\boxed{\frac{d\underline{v}}{dt} = \frac{\partial \underline{v}}{\partial t} + \text{grad}\frac{\underline{v}^2}{2} - \underline{v}\times \text{rot}\,\underline{v} = \underline{g} - \frac{1}{\rho}\text{grad}\,\underline{p}}$$

If $\rho = \rho(p)$, $-\frac{1}{\rho(p)}\text{grad}\,p = -\text{grad}\int_{p_0}^{p}\frac{dp}{\rho(p)}$

If ρ = const. the unknown variables are: v_x, v_y, v_x, p

$$\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z} = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z} = g_z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

11. Bernoulli equation

Inviscid flow: $\boxed{\mu = 0}$

$$\underbrace{\int_1^2 \frac{\partial \underline{v}}{\partial t}d\underline{s}}_{I} + \underbrace{\int_1^2 \text{grad}\frac{v^2}{2}d\underline{s}}_{II} - \underbrace{\int_1^2 \underline{v}\times\text{rot}\,\underline{v}\,d\underline{s}}_{III} = \underbrace{\int_1^2 \underline{g}\,d\underline{s}}_{IV} - \underbrace{\int_1^2 \frac{1}{\rho}\text{grad}\,p\,d\underline{s}}_{V}$$

a) Since $\int_1^2 \text{grad}\,f\,d\underline{s} = f_2 - f_1$ integral II $= \frac{v_2^2 - v_1^2}{2}$

b) If $\underline{g}$ = - gradU integral IV = – (U_2– U_1)

c) In case of steady flow ($\frac{\partial \underline{v}}{\partial t} = \underline{0}$) integral I =0

d) intergral III = 0, if

- $\underline{v}$ = 0 static fluid
- rot$\underline{v}$ = 0 potential flow
- d$\underline{s}$ lies in the plane determined by $\underline{v}$ and rot$\underline{v}$ vectors
- d$\underline{s}$ || $\underline{v}$ integration along streamlines
- d$\underline{s}$ || rot$\underline{v}$ integration along vortex lines

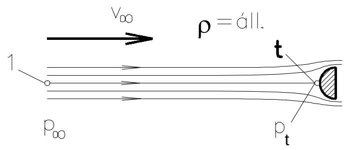
e) If $\rho = \text{const.}$ integral V = $-\frac{p_2 - p_1}{\rho}$, if $\rho = \rho(p)$, integral V = $\int_{p_1}^{p_2} \frac{dp}{\rho(p)}$

In case of inviscid, steady flow of incompressible fluid ($\rho = \text{const.}$), if $\underline{g}$ = - gradU and integration along streamlines:

$$\boxed{\frac{v_1^2}{2} + \frac{p_1}{\rho} + U_1 = \frac{v_2^2}{2} + \frac{p_2}{\rho} + U_2}$$

The Bernoulli's sum = const. along streamlines.

12. Static, dynamic and total pressure



$$\frac{v_1^2}{2} + \frac{p_1}{\rho} + U_1 = \frac{v_2^2}{2} + \frac{p_2}{\rho} + U_2$$

In stagnation point $\underline{v}$ = 0, so $p_\infty + \frac{\rho}{2} v_\infty^2 = p_t$

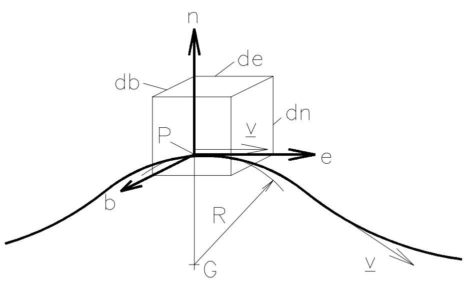
$p_d = \frac{\rho}{2} v_\infty^2$ dynamic pressure

p_∞ static pressure

p_t total, stagnation pressure

Bernoulli equation in case of inviscid, steady flow of incompressible fluid, disregarding the field of force: the total pressure is constant along streamlines.

13. Euler equation in streamwise ("natural") co-ordinate-system



Steady flow of inviscid (μ = 0) fluid. **e** coordinate is tangent to the streamline, **n** is normal to it and cross the center of curvature, A **b** binormal coordinate perpendicular to **e** and **n**.

<u>In **e** direction</u>

Force acting on differential fluid particle of edge length db, dn and de (mass: $dm = \rho dbdnde$) in e direction:

$$dF_e = p\ db\ dn - \left[p + \left(\frac{\partial p}{\partial e} \right) de \right] db\ dn + \rho\ db\ dn\ de\ g_e\ ,$$

where g_e the e component of the field of force.

Since the flow is steady only convective acceleration exists, and $v_n = v_b = 0 \quad a_{conv} = v\frac{\partial v}{\partial e}$.

$$\rho dbdndev\frac{\partial v}{\partial e} = -\frac{\partial p}{\partial e} de\ db\ dn + \rho\ db\ dn\ de\ g_e\ .$$

$$\boxed{v\frac{\partial v}{\partial e} = -\frac{1}{\rho}\frac{\partial p}{\partial e} + g_e}$$

<u>In **n** direction</u>

$dm\frac{v^2}{R}$ centripetal force is needed to move dm mass with v velocity along a streamline of a radius of curvature R:

$$-\rho dedbdn\frac{v^2}{R} = pdbde - \left[p + \left(\frac{\partial p}{\partial n} \right) dn \right] dbde + \rho dedbdn g_n$$

$$\boxed{-\frac{v^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} + g_n}$$

<u>In **b** direction</u>

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial b} + g_b$$

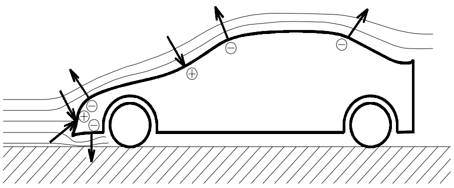
In normal co-ordinate direction, disregarding g:

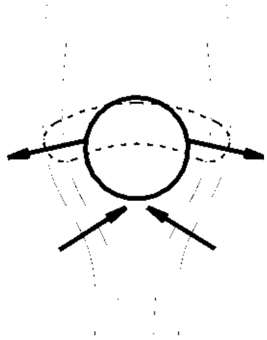
$$\boxed{\frac{v^2}{R} = \frac{1}{\rho}\frac{\partial p}{\partial n}}$$

Consequences:

a) if the streamlines are parallel straight lines (R=∞) the pressure doesn't change perpendicular to the streamlines,

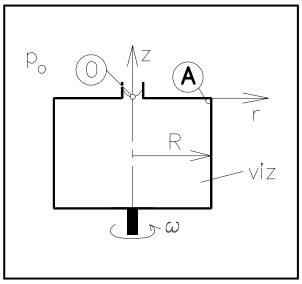
b) if the streamlines are curved the pressure changes perpendicular to the streamlines: it increases outwards from the center of curvature.





raindrop

14. Rotating tank



Forced vortex in absolute system, $\omega[1/s]$ angular velocity, $p_A - p_0 = ?$
3 different ways of solution:

a) co-rotating co-ordinate system: hydrostatics
b) absolute system Bernoulli equation;
c) absolute system, Euler equation in streamwise ("natural") co-ordinate-system

a) $p_A - p_0 = -\rho(U_A - U_0) = -\rho\left(-\frac{R^2\omega^2}{2} - 0\right) = \rho\frac{R^2\omega^2}{2}$

b) $\int_0^A \frac{\partial \underline{v}}{\partial t} d\underline{s} + \int_0^A \text{grad}\frac{v^2}{2} d\underline{s} - \int_0^A \underline{v} \times \text{rot}\underline{v} d\underline{s} = \int_0^A \underline{g} d\underline{s} - \int_0^A \frac{1}{\rho}\text{grad}p d\underline{s}$.

I II III IV V

Steady flow, integral I =0, integral II = $\left(v_A^2 - v_0^2\right)/2$, integral III ≠ 0 since $\text{rot}\underline{v} \neq 0$, and no streamline connects points O and A. Since $\underline{g} \perp d\underline{s}$ integral IV =0, ρ = const. integral

$V = -(p_A - p_0)/\rho$ $\quad p_A - p_0 = \rho\int_0^A \underline{v} \times \text{rot}\underline{v} d\underline{s} - \rho\frac{v_A^2 - v_0^2}{2}$. $v = \omega r$ and $(\text{rot}\underline{v})_z = dv/dr + v/r$

$\Rightarrow (\text{rot}\underline{v})_z = 2\omega$. $\underline{v}$, $\text{rot}\underline{v}$ and $d\underline{s}$ vectors are perpendicular to each-other, and $|d\underline{s}| = dr$, furthermore $v_A = R\omega$ and $v_0 = 0$:

$$p_A - p_0 = \rho\int_0^R (r\omega)2\omega dr - \rho\frac{R^2\omega^2}{2} = \rho R^2\omega^2 - \rho\frac{R^2\omega^2}{2} = \rho\frac{R^2\omega^2}{2}.$$

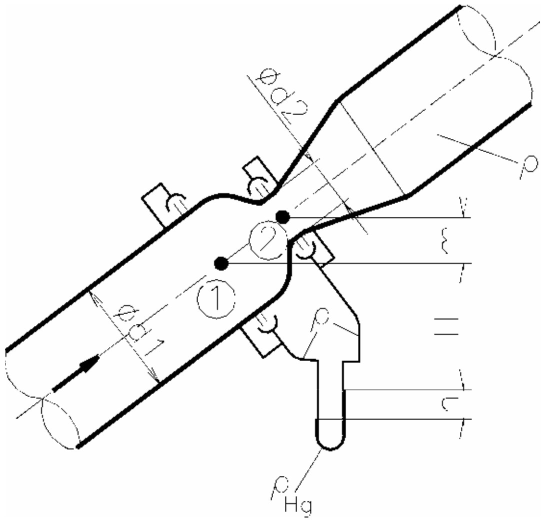
c)

$$\frac{v^2}{R} = \frac{1}{\rho}\frac{\partial p}{\partial n} - g_n.$$

R = r, dn = dr (streamlines are concentric circles), $g_n = 0$

$$\int_{p_0}^{p_A} dp = \int_0^R \rho\frac{v^2}{r} dr = \int_0^R \rho r\omega^2 dr \Rightarrow p_A - p_0 = \rho\frac{R^2\omega^2}{2}.$$

15. Measurement of flow rate by using Venturi meter



h [m]= f(q_v) = ? ρ and ρ_M density of water and mercury

$$\frac{v_1^2}{2}+\frac{p_1}{\rho}+U_1=\frac{v_2^2}{2}+\frac{p_2}{\rho}+U_2$$

U-tube manometer: $p_1+\rho g(H+h)=p_2+\rho g(m+H)+\rho_{Hg}gh \quad p_1-p_2=(\rho_{Hg}-\rho)gh+\rho gm$.

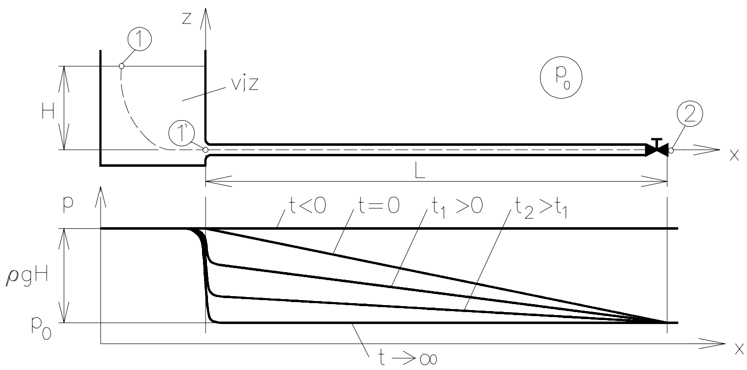
$$\frac{v_2^2}{2}-\frac{v_1^2}{2}=\frac{v_1^2}{2}\left[\left(\frac{v_2^2}{v_1^2}\right)^2-1\right]=\frac{p_1-p_2}{\rho}-gm=\frac{\rho_{Hg}-\rho}{\rho}gh$$

continuity equation: $v_1A_1 = v_2A_2 \Rightarrow v_2/v_1 = (d_1/d_2)^2$

$$v_1=\sqrt{\frac{\left(\frac{\rho_{Hg}}{\rho}-1\right)2gh}{\left(\frac{d_1}{d_2}\right)^4-1}}$$

Flow rate: $\boxed{q_v=\frac{d_1^2\pi}{4}v_1=K\sqrt{h}}$

16. Unsteady discharge of water from a tank



$$\int_1^2 \frac{\partial \underline{v}}{\partial t} d\underline{s} + \int_1^2 \mathrm{grad}\frac{v^2}{2} d\underline{s} - \int_1^2 \underline{v}\times \mathrm{rot}\underline{v} d\underline{s} = \int_1^2 \underline{g} d\underline{s} - \int_1^2 \frac{1}{\rho}\mathrm{grad}p\, d\underline{s}$$

I II III IV V

$$\int_1^2 \frac{\partial \underline{v}}{\partial t} d\underline{s} + \left[\frac{v^2}{2} + \frac{p}{\rho} + U\right]_1^2 = 0$$

In point 1 $p_1 = p_0$, $z = H$, v = 0. In point 2 $p_2 = p_0$. $z = 0$, the velocity is $v_2 = v(t)$.

$$\int_1^2 \frac{\partial \underline{v}}{\partial t} d\underline{s} = \int_1^{1'} \frac{\partial \underline{v}}{\partial t} d\underline{s} + \int_{1'}^2 \frac{\partial \underline{v}}{\partial t} d\underline{s}\,.$$

where $\partial \underline{v}/\partial t$ acceleration vector $|\partial \underline{v}/\partial t|$ is indicated by a, $\partial \underline{v}/\partial t \,||\, d\underline{s}$, $\partial \underline{v}/\partial t$ and $d\underline{s}$ point at the same direction.

$$v_1 A_1 = v_2 A_2 \Rightarrow a_1 A_1 = a_2 A_2$$

$$\int_1^2 \frac{\partial \underline{v}}{\partial t} d\underline{s} = \int_{1'}^2 a ds = aL$$

$$\frac{dv}{dt}L + \frac{v^2}{2} + \frac{p_0}{\rho} = \frac{p_0}{\rho} + gH$$

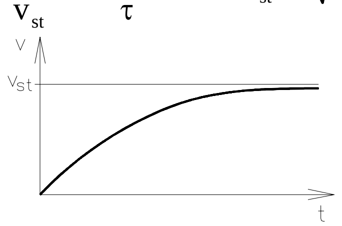
In case of steady flow $(\frac{dv}{dt} = 0)$ $\frac{v_{st}^2}{2} = gH$

$$\frac{dv}{v_{st}^2 - v^2} = \frac{dt}{2L}.$$

$$\int_0^{v/v_{st}} \frac{d\frac{v}{v_{st}}}{1-\left(\frac{v}{v_{st}}\right)^2} = \frac{v_{st}}{2L}\int_0^t dt\,.$$

$$\mathrm{arth}\frac{v}{v_{st}} = \frac{t v_{st}}{2L} \quad \tau = \frac{2L}{v_{st}}$$

$\frac{v}{v_{st}} = \mathrm{th}\frac{t}{\tau}$ where $v_{st} = \sqrt{2gH}$.

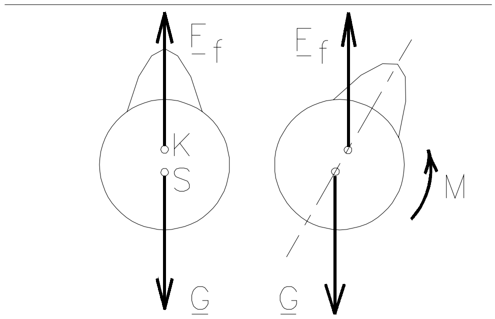


17. Floating of bodies

Body volume: ΔV, pressure distribution is characterized by $\mathrm{grad}p$, Pressure force: $\Delta\underline{F} \cong -\mathrm{grad}\,p\,\Delta V$ $\Delta\underline{F} = -\rho\,\underline{g}\,\Delta V$. In gravitational field buoyant force = weight of the volume displacement. The buoyant force vector crosses the center of displaced volume.

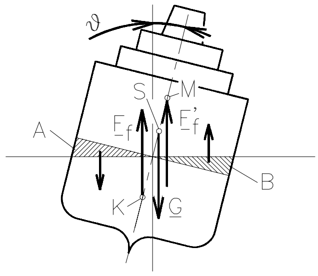
The body is floating if the average density is equal or less than the density of fluid.

Stability of floating body: submarines and ships.



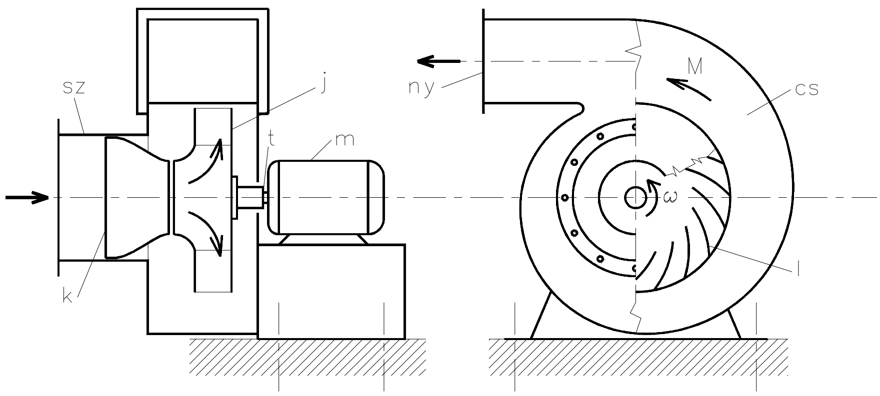
If the center of gravity S is lower than the center of displaced volume K, a moment M arises, decreasing the angle of deflection.

If S center of gravity is above the center of displaced volume K a moment is arising to a certain angle of deflection decreasing the angle of deflection.



At deflection the position, magnitude of weight and magnitude of buoyant force does not change. The line of application of buoyant force displaces. As a consequence of the deflection a wedge-shaped part of the body (A) emerges from the water and the B part of body sinks. So a couple of forces arise, displacing the buoyant force vector. The new line of application crosses the symmetry plane in point M (metacenter). If S is under metacenter M the ship is in stable equilibrium state.

18. Radial-flow fan, Euler equation for turbines

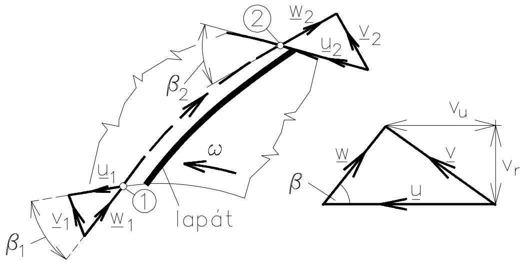


sz: inlet, k: suction nozzle, j: impeller, l: blades, cs: casing, ny: outlet, t: shaft, m: electric motor, M: moment, ω: angular velocity.
Task: increase of total pressure of gas:

$$\Delta p_t = p_{ot} - p_{it} = \left(p + \frac{\rho}{2}v^2\right)_o - \left(p + \frac{\rho}{2}v^2\right)_i$$

available performance: $\boxed{P = q_v \Delta p_t}$, where q_v [kg/m[3]] is the flow rate.

Bernoulli equation in relative coordinate-system (steady flow of incompressible and inviscid fluid) between points 1 and 2 of the same streamline.:



$$\int_1^2 \frac{\partial \underline{w}}{\partial t} d\underline{s} + \frac{w_2^2 - w_1^2}{2} - \int_1^2 \underline{w} \times \mathrm{rot}\,\underline{w}\, d\underline{s} = \int_1^2 \underline{g}\, d\underline{s} - \int_1^2 \frac{1}{\rho} \mathrm{grad}\,p\; d\underline{s}\,.$$

I II III IV V

$$\underline{g} = \underline{g}_c + \underline{g}_{Cor} = -\mathrm{grad}(U_g + U_c) + 2\underline{w} \times \underline{\omega},\; U_g \cong 0,\; U_c = -\frac{r^2\omega^2}{2}$$

$$\int_1^2 \underline{g}\, d\underline{s} = \left(\frac{r_2^2\omega^2}{2} - \frac{r_1^2\omega^2}{2}\right) + \int_1^2 2\underline{w} \times \underline{\omega}\, d\underline{s}\,,$$

Since $\underline{v} = \underline{w} + \underline{u}$, if $\mathrm{rot}\,\underline{v} = \underline{0} \Rightarrow \mathrm{rot}\,\underline{w} = -\mathrm{rot}\,\underline{u}$. Since $|\underline{u}| = r\omega$ $\mathrm{rot}\,\underline{u} = 2\underline{\omega}$.

$$-\int_1^2 \underline{w} \times \mathrm{rot}\,\underline{w}\, d\underline{s} = -\int_1^2 \underline{w} \times (-2\underline{\omega})\, d\underline{s} = \int_1^2 2\underline{w} \times \underline{\omega}\, d\underline{s}$$

Finally:

$$\frac{w_1^2}{2} + \frac{p_1}{\rho} - \frac{r_1^2\omega^2}{2} = \frac{w_2^2}{2} + \frac{p_2}{\rho} - \frac{r_2^2\omega^2}{2}$$

$$\underline{w} = \underline{v} - \underline{u} \Rightarrow w^2 = v^2 + u^2 - 2\underline{u}\,\underline{v}$$

$$\frac{v_2^2}{2} + \frac{u_2^2}{2} - \underline{v}_2\underline{u}_2 - \frac{r_2^2\omega^2}{2} - \frac{v_1^2}{2} - \frac{u_1^2}{2} + \underline{v}_1\underline{u}_1 + \frac{r_1^2\omega^2}{2} + \frac{p_2 - p_1}{\rho} = 0.$$

$$u_1 = r_1\omega$$

$$\Delta p_t = p_{2t} - p_{1t} = \left(p_2 + \frac{\rho}{2}v_2^2\right) - \left(p_1 + \frac{\rho}{2}v_1^2\right) = \rho(\underline{v}_2\underline{u}_2 - \underline{v}_1\underline{u}_1).$$

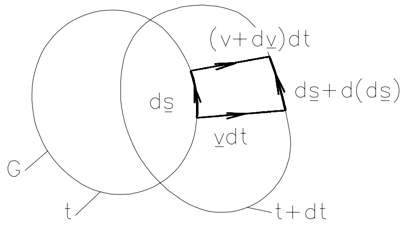
$$\underline{v}_2\,\underline{u}_2 = v_{2u}\,u_2\,,$$

$$\boxed{\Delta p_{t\,id} = \rho(v_{2u}u_2 - v_{1u}u_1)}$$

If $\underline{v}_{1u} = 0 \Rightarrow \quad \Delta p_{t\,id} = \rho v_{2u} u_2\,.$

19. Theorems for vorticity: Thomson' and Helmholtz' theorems

Thomson' theorem (inviscid fluid)

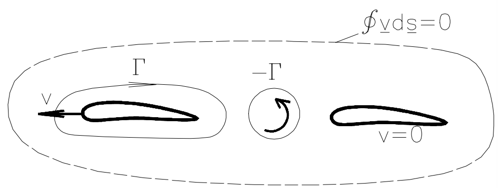


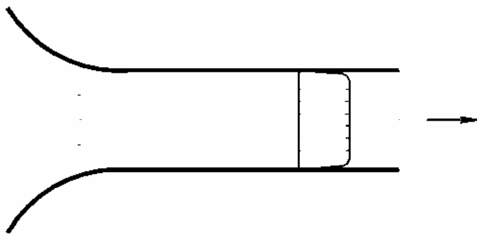
Circulation: $\Gamma = \oint_G \underline{v}\, d\underline{s}$. Temporal change of circulation along closed fluid line $\frac{d\Gamma}{dt} = \frac{d}{dt}\oint_G \underline{v}\, d\underline{s} = ?$ If $\underline{g} = -\mathrm{grad}U$ and ρ=const. or ρ=ρ(p), by using Euler equation:

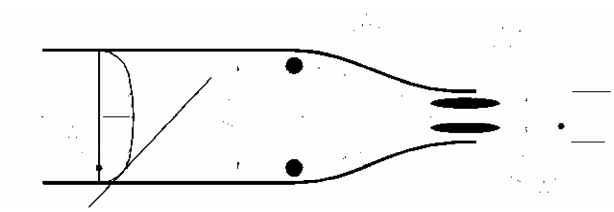
$$\boxed{\frac{d}{dt}\oint_G \underline{v} d\underline{s} = 0}$$

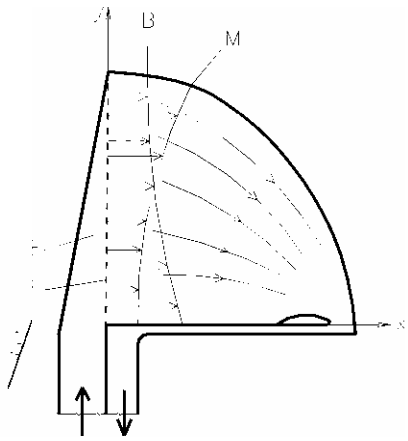
In flow of incompressible and inviscid fluid in potential field of force no vorticity arises.

Applications:
Starting and stopping vortex (vortex shedding), making velocity distribution uniform, flow in water reservoir





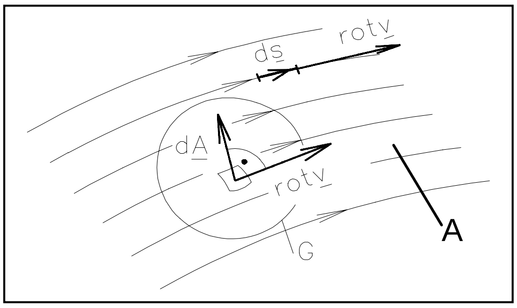




$\Gamma = \oint_G \underline{v} d\underline{s} = \int_A \mathrm{rot}\underline{v} d\underline{A}$. $\frac{(\mathrm{rot}\,\underline{v})_{\vartheta 2}}{(\mathrm{rot}\,\underline{v})_{\vartheta 1}} = \frac{\Delta A_1}{\Delta A_2} = \frac{D_2}{D_1}$. $(\mathrm{rot}\underline{v})_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0$. $\partial v_y/\partial x < 0 \Rightarrow \partial v_x/\partial y < 0$.

Helmholtz' I. theorem $\boxed{\mu = 0}$

Fluid vortex line: rot $\underline{v} \times d\underline{s} = 0$, fluid vortex surface: rot $\underline{v} \times d\underline{A} = 0$

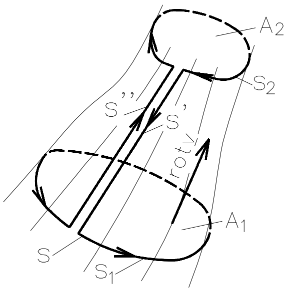


Since $\frac{d}{dt}\oint_G \underline{v}d\underline{s} = 0$, a flowing vortex surface remains vortex surface.
Two vortex sheets intersect each other along a vortex line.

A flowing vortex line, which can be regarded as line of intersection of two flowing vortex surfaces, consists of the same fluid particles.

Consequence: The vortex in smoke ring or in cloud of smoke emerging from a chimney preserves the smoke.

Helmholtz' II. theorem



Flowing vortex tube

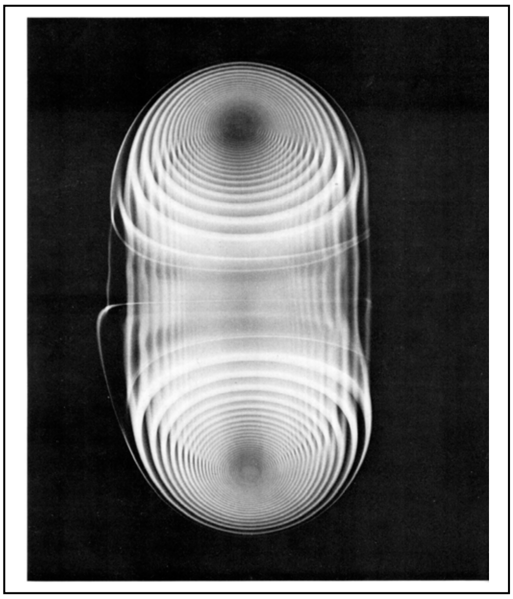
$$\oint_S \underline{v}d\underline{s} = \oint_{S_1} \underline{v}d\underline{s} + \oint_{S_2} \underline{v}d\underline{s} = 0$$

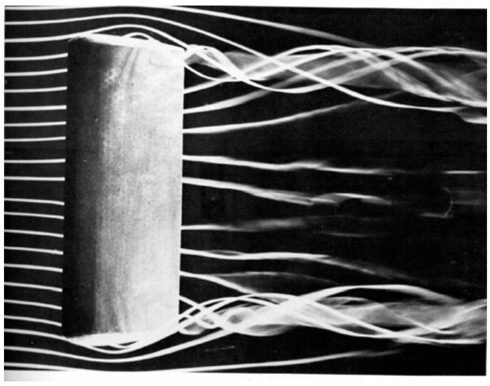
$$\oint_{S_1} \underline{v}d\underline{s} = \oint_{S_2} \underline{v}d\underline{s},$$

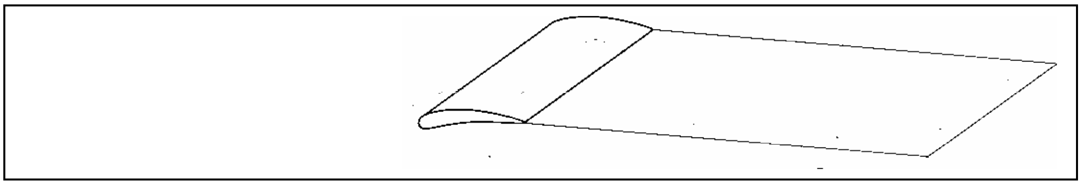
$$\int_{A_1} rot\underline{v}d\underline{A} = \int_{A_2} rot\underline{v}d\underline{A}.$$

$\int_A rot\underline{v}d\underline{A}$ is constant over all cross sections along a vortex tube and it does not change temporally.

Consequences: the vortex tube is either a closed line (a ring) or ends at the boundary of the flow field. A $\Rightarrow$ 0 rot$\underline{v}$ $\Rightarrow \infty$.







Induced vortex, tip vortex of finite airfoil.

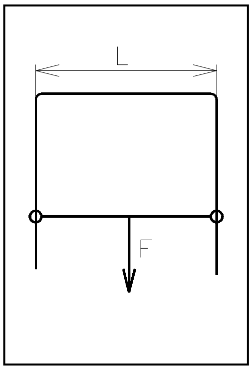
Flight of wild-gooses in V shape.

Vortex in tub after opening the sink.

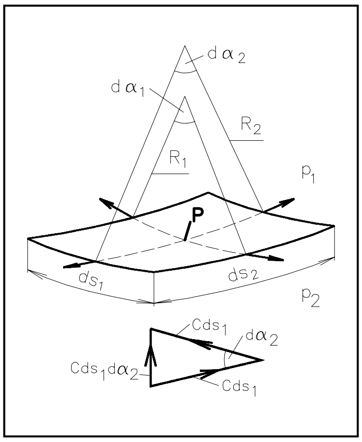
Tornado

20.Pressure measurements

Surface tension



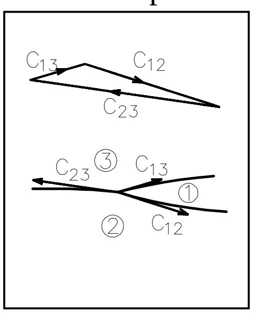
$F = 2LC$ C [N/m] surface tension coefficient. For water air combination $C = 0.025[N/m]$.



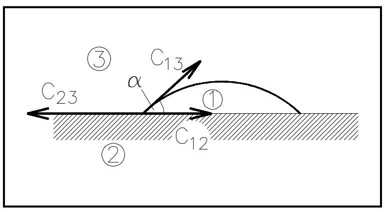
$(p_1 - p_2) ds_1 ds_2 = C ds_1 d\alpha_2 + C ds_2 d\alpha_1$.

$$\Delta p = p_1 - p_2 = C\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

In case of spheres $R_1 = R_2 = R \Rightarrow \Delta p = 2C/R$, and bubbles: $\Delta p = 4C/R$



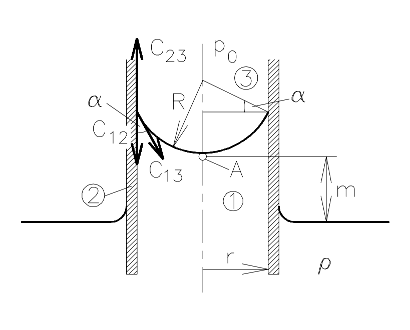
If $|C_{23}| > |C_{12}| + |C_{13}|$, fluid 1 expands on the surface of fluid 2 (e.g. oil on water.



$C_{23} ds = C_{12} ds + C_{13} \cos\alpha ds$.

$\cos\alpha = (C_{23} - C_{12})/C_{13}$. $C_{23} > C_{12} \Rightarrow \alpha < 90°$, $\alpha > 90°$ (mercury) Ha $|C_{23}| > |C_{12}| + |C_{13}|$, the fluid expands over the surface of solid body petroleum gets out of open bottle.

Capillary rise



$$p_0 - p_A = 2C_{13}/R = 2C_{13}\cos\alpha / r .$$

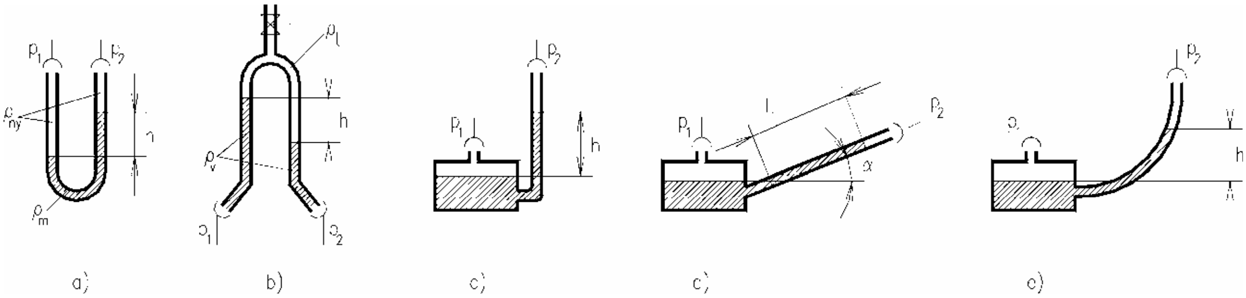
$$p_0 - p_A = \rho g m$$

$$m = \frac{2C_{13}}{\rho g r}\cos\alpha$$

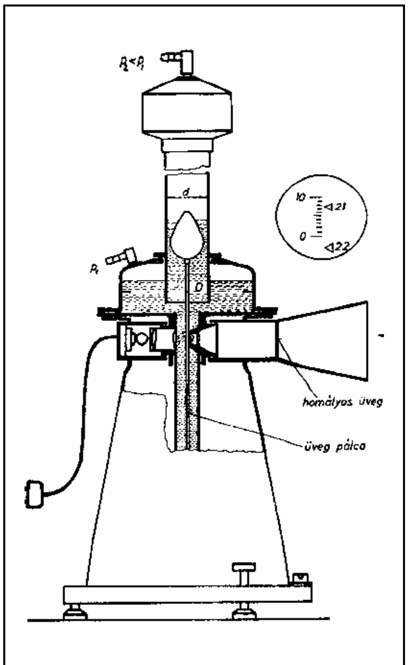
In case of mercury capillary drop.

Measurement of pressure

Manometers (for measuring pressure differences)
Micromanometers: U-tube manometer $p_1 - p_2 = (\rho_m - \rho_t)\, g\, h$, "inverse" U-tube manometer $p_1 - p_2 = (\rho_w - \rho_a)\, g\, h$, inclined tube manometer $L = H/\sin\alpha$, relative error: $e = \Delta s/L = (\Delta s/H)\sin\alpha$, bent tube manometer (e = const.):



Betz-micromanometer



Pressure taps