Turbulence modelling II.

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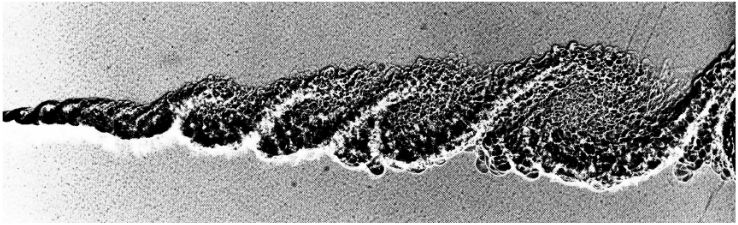
Many scales of turbulence

Density variation visualise the different scales of turbulence in a mixing layer



Goal: Try to find some rules about the properties of turbulence at different scales

Kinetic energy

Kinetic energy:

$$E \stackrel{\text{def}}{=} \frac{1}{2} u_i u_i \tag{1}$$

Its Reynolds decomposition:

$$E = \frac{1}{2} u_i u_i = \frac{1}{2} (\overline{u_i}\,\overline{u_i} + 2u_i'\overline{u_i} + u_i'u_i') \tag{2}$$

Its Reynolds average

$$\overline{E} = \underbrace{\frac{1}{2} (\overline{u_i}\,\overline{u_i})}_{\hat{E}} + \underbrace{\frac{1}{2} (\overline{u_i'u_i'})}_{k} = \hat{E} + k \tag{3}$$

- The kinetic energy of the mean flow: $\hat{E}$
- The kinetic energy of the turbulence: k (Turbulent Kinetic Energy, TKE)

Richardson energy cascade
The poem

Lewis Fry Richardson (1920):

“Big whirls have little whirls,
that feed on their velocity;
and little whirls have lesser whirls,
and so on to viscosity.”

“Nagy örvény kisebbet plántál,
melyet sebességével táplál;
majd az még kisebbet szülvén,
viszkozításba tűnik szürkén.”

Richardson energy cascade

Vortex scales

High Re flow is investigated

- Typical velocity of the flow $\mathcal{U}$
- Typical length scale of the flow $\mathcal{L}$
- Corresponding Reynolds number ($\mathcal{R}e = \frac{\mathcal{U}\mathcal{L}}{\nu}$) is high

Turbulence is made of vortices of different sizes

Each class of vortex has:

- length scale: l
- velocity scale: $u(l)$
- time scale: $\tau(l) = l/u(l)$

Richardson energy cascade

The big scales

Biggest vortices

- size $l_0 \sim \mathcal{L}$
- velocity $u_0 = u_0(l_0) \sim u' = \sqrt{2/3k} \sim \mathcal{U}$

$\Rightarrow Re = \frac{u_0 l_0}{\nu}$ is also high

Fragmentation of the big vortices

- High Re corresponds to low viscous stabilisation
- Big vortices are unstable
- Big vortices break up into smaller ones

Richardson energy cascade

To the small scales

Inertial cascade

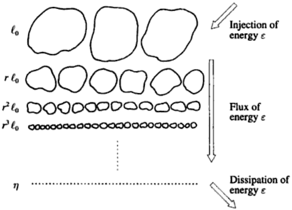
- As long as $Re(l)$ is high, inertial forces dominate, the break up continues
- At small scales $Re(l) \sim 1$ viscosity starts to be important
  - The kinetic energy of the vortices dissipates into heat

Richardson energy cascade

Connection between small and large scales

Dissipation equals production

- Dissipation is denoted by ε
- Because of the cascade can be characterised by large scale motion
- Dissipation: $\varepsilon \sim \frac{\text{kin. energy}}{\text{timescale}}$ at the large scales
  - By formula: $\varepsilon = \frac{u_0^2}{l_0/u_0} = \frac{u_0^3}{l_0}$



NS symbol

For the description of development rules, it is useful to define the following NS symbol:

$$NS(u_i) \stackrel{\text{def}}{=} \partial_t u_i + u_j \partial_j u_i = \underbrace{-\frac{1}{\rho}\partial_i p + \nu \partial_j s_{ij}}_{\partial_j t_{ij}} \tag{4}$$

where: $s_{ij} \stackrel{\text{def}}{=} \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is the deformation (rate of strain) part of the derivative tensor $\partial_j u_i$.

Transport equation of k

Definitions 2

Let us repeat the development of the Reynolds equation!

$$\overline{NS(\overline{u_i} + u_i')} \tag{5}$$

$$\partial_t \overline{u_i} + \overline{u_j}\,\partial_j \overline{u_i} = \partial_j \underbrace{\left[-\frac{1}{\rho}\overline{p}\,\delta_{ij} + \nu \overline{s}_{ij} - \overline{u_i' u_j'} \right]}_{\overline{T_{ij}}} \tag{6}$$

The TKE equation

Taking the trace of

$$(NS(u_i) - \overline{NS(u_i)})u'_j + (NS(u_j) - \overline{NS(u_j)})u'_i$$

$$\partial_t k + \overline{u_j}\,\partial_j k = \underbrace{-a_{ij}\overline{s_{ij}}}_{\text{Production}} + \underbrace{\partial_j \left[\overline{u'_j \left(\frac{p'}{\rho} + k' \right)} - \nu \overline{u'_i s'_{ij}} \right]}_{\text{Transport}} - \underbrace{\varepsilon}_{\text{Dissipation}} \tag{7}$$

- Dissipation: $\varepsilon \stackrel{\text{def}}{=} 2\nu \overline{s'_{ij} s'_{ij}}$
- Anisotropy tensor: $a_{ij} \stackrel{\text{def}}{=} \overline{u'_i u'_j} - \frac{1}{3} \underbrace{\overline{u'_l u'_l}}_{2k} \delta_{ij}$

  Deviator part of the Reynolds stress tensor

Production

- Expression: $\mathcal{P} \stackrel{\text{def}}{=} -a_{ij}\overline{s_{ij}}$
- Transfer of kinetic energy from mean flow to turbulence
  - The same term with opposite sign in the equation for kin. energy of mean flow
- The mechanism to put energy in the 'Richardson' cascade
- Happens at the large scales

Dissipation

- Expression: $\varepsilon \stackrel{\text{def}}{=} 2\nu\overline{s'_{ij}s'_{ij}}$
- Conversion of kinetic energy of turbulence to heat
  - Work of the viscous stresses at small scale (s'_{ij})
- The mechanism to draw energy from the 'Richardson' cascade
- Happens at the small scales

$\mathcal{P} = \varepsilon$ if the turbulence is homogeneous (isotropic), as in the "Richardson" cascade

The TKE equation

Meaning of the terms (contd.)

Transport

- Expression: $\partial_j \left[\overline{u'_j \left(\frac{p'}{\rho} + k' \right)} - \nu \overline{u'_i s'_{ij}} \right]$
- Transport of turbulent kinetic energy in space
  - The expression is in the form of a divergence $(\partial_j \Box_j)$
  - Divergence can be reformulated to surface fluxes (G-O theorem)

Idea of RANS modelling

- Solving the Reynolds averaged NS for the averaged variables $(\overline{u}, \overline{v}, \overline{w}, \overline{p})$
- The Reynolds stress tensor $\overline{u_i' u_j'}$ is unknown and has to be modelled
- Modelling should use the available quantities $(\overline{u}, \overline{v}, \overline{w}, \overline{p})$

Usefulness

- If the averaged results are useful for the engineers
- i.e. the fluctuation are not interesting "only" their effect on the mean flow
- If modelling is accurate enough

Eddy Viscosity modell

Idea

- Effect of turbulence is similar to effect of moving molecules in kinetic gas theory
- The exchange of momentum between layers of different momentum is by the perpendicularly moving molecules
- Viscous stress is computed by: $\Phi_{ij} = 2\nu S_{ij}$

Eddy Viscosity model (contd.)

In equations...

- Only the deviatoric part is modelled
- The trace (k) can be merged to the pressure (modified pressure), and does not need to be modelled
- Modified pressure is used in the pressure correction methods to satisfy continuity (see Poisson eq. for pressure)

$$\overline{u_i'u_j'} - \frac{2}{3}k\delta_{ij} = -2\nu_t\overline{S_{ij}} \tag{8}$$

Eddy Viscosity

Viscosity is a product of a length scale (l') and a velocity fluctuation scale (u')

- The length scale has to be proportional to the distance, what the fluid part moves by keeping its momentum
- The velocity fluctuation scale should be related to the velocity fluctuation caused by the motion of the fluid part

$$\nu_t \sim l'u' \tag{9}$$

Newer results supporting the concept

Coherent structure view of turbulence, proves that there are fluid parts (vortices) which keep their properties for a while, when moving

Two equations models

- Length (l') and velocity fluctuation scales (u') are properties of the flow and not the fluid, they are changing spatially and temporally
- PDE's for describing evolutions are needed

Requirements for the scales

- Has to be well defined
- Equation for its evolution has to be developed
- Has to be numerically 'nice'
- Should be measurable easily to make experimental validation possible

k-e modell

Velocity fluctuation scale

- TKE is characteristic for the velocity fluctuation
- It is isotropic (has no preferred direction)

$$u' \sim \sqrt{k} \tag{10}$$

Length scale

- Integral length scale is well defined (see correlations)
- No direct equation is easy to develop
- Length scale is computed through the dissipation

Recall: $\varepsilon = \frac{u_0^3}{l_0} \Rightarrow l' \sim \frac{k^{3/2}}{\varepsilon}$

Equation for the eddy viscosity

$$\nu_t = C_\nu \frac{k^2}{\varepsilon} \tag{11}$$

C_ν is a constant to be determined by theory or experiments...

Our status...?

- We have two unknowns (k, ε) instead of one (ν_t)

k model equation

Equation for k was developed, but there are unknown terms:

$$\partial_t k + \overline{u_j}\,\partial_j k = \underbrace{-a_{ij}\overline{s_{ij}}}_{\text{Production}} + \underbrace{\partial_j \left[\overline{u_j'\left(\frac{p'}{\rho} + k'\right)} - \nu \overline{u_i' s_{ij}'} \right]}_{\text{Transport}} - \underbrace{\varepsilon}_{\text{Dissipation}} \tag{12}$$

Production

Production is directly computable, by using the eddy viscosity hypothesis

$$\mathcal{P} = -a_{ij}\overline{S_{ij}} = 2\nu_t \overline{S_{ij}}\,\overline{S_{ij}} \tag{13}$$

k model equation

Dissipation

Separate equation will be derived

Transport $\partial_j T_j$

- Can be approximated by gradient diffusion hypothesis

$$T_j = \frac{\nu_t}{\sigma_k} \partial_j k \tag{14}$$

- σ_k is of Schmidt number type to rescale ν_t to the required diffusion coeff.
  - To be determined experimentally

Summarised k model equation

$$\partial_t k + \overline{u_j}\,\partial_j k = 2\nu_t \overline{S_{ij}}\,\overline{S_{ij}} - \varepsilon - \partial_j\left(\frac{\nu_t}{\sigma_k}\partial_j k\right) \tag{15}$$

- Everything is directly computable (except ε)
- The LHS is the local and convective changes of k
  - Convection is an important property of turbulence (it is appropriately treated by these means)

Model equation for ε

- It is assumed that it is described by a transport equation
- Instead of derivation, based on the k equation

$$\partial_t \varepsilon + \overline{u_j}\, \partial_j \varepsilon = C_{1\varepsilon} \mathcal{P} \frac{\varepsilon}{k} - C_{2\varepsilon} \varepsilon \frac{\varepsilon}{k} - \partial_j \left(\frac{\nu_t}{\sigma_\varepsilon} \partial_j \varepsilon \right) \qquad (16)$$

- Production and dissipation are rescaled ($\frac{\varepsilon}{k}$) and 'improved' by constant coefficients ($C_{1\varepsilon}$, $C_{2\varepsilon}$)
- Gradient diffusion for the transport using Schmidt number of σ_ε
- The ε equation is not very accurate! :)

Constants of the standard k-e model

$$C_\nu = 0.09 \tag{17}$$
$$C_{1\varepsilon} = 1.44 \tag{18}$$
$$C_{2\varepsilon} = 1.92 \tag{19}$$
$$\sigma_k = 1 \tag{20}$$
$$\sigma_\varepsilon = 1.3 \tag{21}$$

Example for the constants

Homogeneous turbulence

$$\mathrm{d}_t k = \mathcal{P} - \varepsilon \tag{22}$$

$$\mathrm{d}_t \varepsilon = C_{1\varepsilon}\mathcal{P}\frac{\varepsilon}{k} - C_{2\varepsilon}\varepsilon\frac{\varepsilon}{k} \tag{23}$$

Example for the constants

Decaying turbulence

Since $\mathcal{P} = 0$ the system of equations can be solved easily:

- $k(t) = k_0 \left(\frac{t}{t_0} \right)^{-n}$
- $\varepsilon(t) = \varepsilon_0 \left(\frac{t}{t_0} \right)^{-n-1}$
- $n = \frac{1}{C_{2\varepsilon} - 1}$
- n is measurable 'easily'

k-ω modell

- k equation is the same
- $\omega \stackrel{\text{def}}{=} \frac{1}{C_\nu}\frac{\varepsilon}{k}$ Specific dissipation, turbulence frequency (ω)
- equation for ω similarly to ε equation
  - transport equation, with production, dissipation and transport on the RHS
- ω equation is better close to walls
- ε equation is better at far-field

$\Rightarrow$ SST model blends the two type of length scale equations, depending on the wall distance

Required Boundary Conditions

The turbulence model PDE's are transport equations, similar to the energy equation

- Local change
- Convection
- Source terms
- Transport terms

Inlet Boundary Conditions

- Neumann or Dirichlet or mixed type of BC can be used generally
- Inlet is usually Dirichlet (specified value)

Final goal

- How to prescribe k and ε or ω at inlet boundaries?

Approximation of inlet BC's

Turbulence intensity

To use easy quantities, which can be guessed

Develop equations to compute k and ε or ω from quantities, which can be guessed by engineers

Turbulence intensity

$$Tu \overset{\text{def}}{=} \frac{u'}{\overline{u}} = \frac{\sqrt{2/3k}}{\overline{u}}$$

Length scale

$l' \sim \frac{k^{3/2}}{\varepsilon} \Rightarrow \varepsilon$

- From measurement (using Taylor hypothesis)
- Law of the wall (later)
- Guess from hydraulic diameter $l \approx 0.07 d_H$

Importance of inlet BC's

If turbulence is governing a flow

- Example: Atmospheric flows, where geometry is very simple (flat land, hill) turbulence is complex
  - by spatial history of the flow
  - over rough surface
  - including buoyancy effects
- Sensitivity to turbulence at the inlet has to be checked
  - the uncertainty of the simulation can be recognised
  - measurement should be included
  - the simulation domain should be extended upstream

Questions?

Thanks for your attention!