<u>**Technical Acoustics and Noise Control**</u> *(lecture notes for self-learning)*

Dr. Gábor KOSCSÓ titular associate professor (BME Department of Fluid Mechanics)

Lecture 3.

Content:

3.1. The homogeneous linear acoustic wave equation

The direct algebraic relationship of sound field variables is a useful knowledge, but sound is a physical phenomenon, extended in time and space, so in the general modelling of sound field, the space and time dependence of sound field variables (p', v', T' and ρ') must be determined. The space- and time dependent mathematical relationship, the wave acoustic model, is described by the wave equation and its solution, the wave function. To determine the space- and time dependent functions, the space- and time dependent differential equations must be solved, related for the sound phenomenon.

The first steps of mathematical modelling (selection of variables, physical principles, resolution of sound field variables, and simplifications) are the same as in the previous derivation. The difference is the mathematical form of equations expressing physical principles, which are now partial differential equations with place- and time-dependent variables. For simplicity, the derivation is performed in the first step on a plain sound wave propagating in the x direction.

The general 3-dimensional form of the continuity equation,

$$\frac{\partial \rho}{\partial t} + div(\rho \underline{v}) = 0$$

The continuity equation in x direction (let introduce the $v_x = v$ notation),

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$

The resolution of the sound field variables, in a non-moving medium ($v_0 = 0$ m/s),

$$\frac{\partial \rho_0}{\partial t} + \frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x}\left((\rho_0 + \rho')v'\right) = 0$$

The equilibrium value of density is constant in time, so its derivative is zero,

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x}(\rho_0 v' + \rho' v') = 0$$

Furthermore, within the brackets, at the second place, the second order small term is neglected, and after the derivation of the product,

$$\frac{\partial \rho'}{\partial t} + v'\frac{\partial \rho_0}{\partial x} + \rho_0\frac{\partial v'}{\partial x} = 0$$

In a homogeneous medium, the equilibrium density is constant as a function of place, so the second term on the left is zero. The remainder is the linear acoustic continuity equation,

$$\frac{\partial \rho'}{\partial t} + \rho_0\frac{\partial v'}{\partial x} = 0$$

The equation is linear because the unknown sound field variables (and their derivatives) are linear expressions, acoustic because applied neglections are satisfied in sound fields, and it is a continuity equation because the initial equation expresses the principle of mass conservation.

The three-dimensional frictionless equation of motion for fluid flow, the Euler equation, with velocity-derivative tensor

$$\frac{\partial \underline{v}}{\partial t} + \underline{\underline{D}}_v\underline{v} = -\frac{1}{\rho}grad\ p + \underline{g}$$

The Euler equation in x direction (with the v_x= v notation),

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + g_x$$

After the resolution of the sound field variables, in a non-moving medium (v_0= 0 m/s) the equation of motion,

$$\frac{\partial v'}{\partial t} + \frac{\partial v'}{\partial x}v' = -\frac{1}{\rho_0 + \rho'}\frac{\partial (p_0 + p')}{\partial x} + g_x$$

In the case of sound phenomena occurring in mechanical engineering practice, the wavelength is large (in technical normal state air, the wavelength is rounded to one decimal at 50 Hz is 6.9 m, and at 2 kHz is 171.5 mm). Thus, the change in sound field variables (in this case particle velocity) per unit length is small, so the second term on the left side is small in the second order, and negligible with a good approximation. Furthermore, due to the small value of ρ', let $\rho_0+\rho' \approx \rho_0$, so by deriving the sum on the right,

$$\frac{\partial v'}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p_0}{\partial x} + g_x - \frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

On the right side of the equation, the first two terms are the values of the force from the equilibrium pressure and the external force per unit mass (the left side of the x-direction component of the hydrostatic equation), and their signed sum is zero. The disappearance of the external force field in a physical approach means that the air particles float in the air, from an acoustic point of view it means that the presence of a static force field does not affect the sound propagation (e.g. in a gravitational field the sound propagates vertically downwards, upwards and horizontally in the same way). The remaining terms are the linear acoustic equation of motion,

$$\frac{\partial v'}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

The continuity equation and equation of motion include three independent sound field variables (ρ', v' and p'), the solution requires a third independent equation. Writing the place- and time-dependent energy equation does not help in this, because it includes another unknown term the temperature fluctuation (T '). To solve this, we take the third independent equation from the algebraic model (the space and time dependence that is currently important to us are already included in the linear acoustic continuity equation and equation of motion),

$$\frac{p'}{\rho'} = a^2 = \kappa R T_0$$

In the previous equation, the left side is derived from the combination of continuity equation and equations of motion, but the fact that the square of the sound speed is constant is turned out from the equation of energy and the state of gas. In order to reduce the number of variables (elimination of the particle velocity), we differentiate the continuity equation according to time and the equation of motion according to location,

$$\frac{\partial^2 \rho'}{\partial t^2} + \rho_0 \frac{\partial^2 v'}{\partial x \partial t} = 0$$

and

$$\frac{\partial^2 v'}{\partial x \partial t} = -\frac{1}{\rho_0}\frac{\partial^2 p'}{\partial x^2}$$

Taking into account the equality of second-order cross derivatives, substituting the right-hand side of the place-derivative equation of motion for the second term on the left-hand side of the time-derivative continuity equation,

$$\frac{\partial^2 \rho'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = 0$$

From the speed of sound squared expression, using $\rho'=p'/a^2$, the homogeneous linear acoustic wave equation,

$$\frac{1}{a^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = 0$$

Comments:

- The homogeneous linear acoustic wave equation is a second-order, hyperbolic type partial differential equation, a basic equation for describing sound propagation and sound fields. Its significance is given by the physical conclusions that can be drawn from it, its analytical solutions for simple cases and its numerical simulation solutions for complex cases.

- The wave equation was first derived by d'Alembert for other physical phenomena (to describe the mechanical disturbance propagation in strings), but for essentially similar wave propagation, so this is why the literature refers the wave equation as the d'Alembert equation in many places.

- Depending on which variables are eliminated during the derivation, or which variable we replace p' with the linear algebraic relationship, an equation of the same shape as the wave equation can be derived for the other sound field variables (v', T' and ρ'). This seemingly not important, formal mathematical fact physically means that during sound propagation, the sound field variables change simultaneously in space and time.

- In different media, the magnitude of the sound speed is usually between $10^2 \ldots 10^4$, so by rearranging the wave equation, it can be seen that the variability in space is much smaller than in time.

- In other fields of physics, in connection with other phenomena (strings, membrane motion, free-surface fluid motion, optics, electromagnetism,…), of course for other physical variables, but identical shape of differential equation can be derived. Each of these phenomena has a wave nature. Therefore, the equations derived above or of the same shape are called wave equations.

- A three-dimensional wave equation can be derived from the basic three-dimensional equations for the description of general sound spaces in space,

- A three-dimensional wave equation can be derived from the three-dimensional basic equations to describe general three-dimensional sound field,

$$\frac{1}{a^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0$$

Where the Laplace operator,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- The homogeneous acoustic wave equation is suitable for describing the sound field due to simplifications. Considering the wave nature of sound, this itself is a big task. Leaving some simplifications (e.g. no viscosity, heat-insulating medium), an equation similar to the homogeneous wave equation can be derived. The left side of the equations are identical, but the right sides are different. For the homogeneous one the right side is zero. When leaving the simplifications, the right hand side turns to not zero. This equation is called, as inhomogeneous wave equation, with the help of which both sound generation and sound attenuation can be described.

- The homogeneous acoustic wave equation has general and partial solutions. For a specific case, the solution also requires one initial and two boundary conditions.

Solutions of a homogeneous, linear, acoustic wave equation

From an engineering point of view, an equation is worth as much as it can be solved. Therefore, we pay special attention to the different solutions of the wave equation. The solution of the wave equation is called the wave function. In addition to the general solution of the homogeneous wave equation, there are important particular solutions, and we distinguish solutions for free and bounded spaces.

3.2. The general solution of the wave equation

In a homogeneous, continuous medium of infinite extent, the general solution of the wave equation is the following wave function, which describes the one-dimensional plane wave propagation in free space,

$$p'(x,t) = f\left(t - \frac{x}{a}\right) + g\left(t + \frac{x}{a}\right)$$

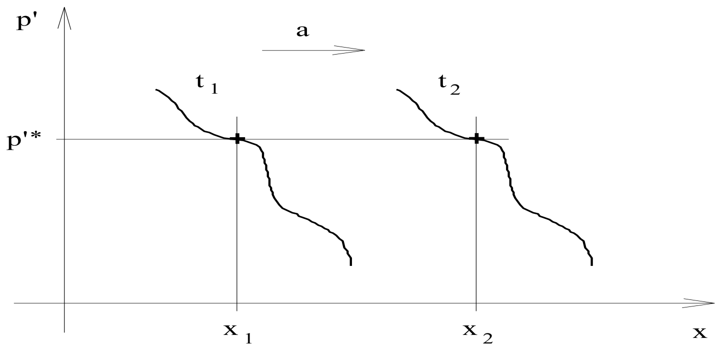
Comments:

- The solution is the sum of two arbitrary functions (f and g) whose arguments, in an unusual way, instead of listing the independent variables (t and x), contain a simple mathematic operation between the time and space (x) variables.

- The formal correctness of the solution can be easily checked. Take component “f” in the first step. The external function is a function of space and time, so substituting it in the wave equation and differentiating twice in time and space the result will be the same. In the case of the space derivative, the double derivative of the external function must be multiplied twice by the derivative of the internal function. For this reason, the shape of the twice-derivatives are the same, their sign is opposite, so the sum is zero, and the solution is correct. A similar result is obtained for the component of the function “g”.

- The f and g functions are twice continuously differentiable, arbitrary functions, so the wave equations do not impose any constraint on the shape of the mechanical disturbance. The physical content of this first astonishing fact is that any arbitrary excitation can create a wave. This is supported by practical observation of the diversity of sounds.

- The argument of the wave function also contains important content. The role in creating the formal correctness of the solution has been seen before, now let's look at what the physical content of the argument is. In the first step, we examine the component of the "f" function, and the following figure will help our understanding.



Distribution of the sound pressure of a wave moving in the positive x direction as a function of space at t_1 and t_2 time

Select the point on the wave front marked with a star at x_1 place and at t_1 time. At this point, the sound pressure is p'*. The point marked with a star on the wave front moves a little later, at time t_2, from left to right, to point x_2, due to the wave nature of the sound. In the case of plane wave sound propagation, the sound rays do not diverge and do not converge. Furthermore, due to the initial simplifications, during the sound propagation there is no loss (no attenuation) and no creation of sound (no sound source), so at the new time t_2 and place x_2 in the point of the wave front, marked with star, the sound pressure is the same as it was in the origin (x_1, t_1). Returning to the solution function,

$p'^*(x_1, t_1) = p'^*(x_2, t_2)$ in our case, examining component "f", $f\left(t_1 - \frac{x_1}{a}\right) = f\left(t_2 - \frac{x_2}{a}\right)$

For any function "f", equality exists if we substitute the same number in "f",,

$t_1 - \frac{x_1}{a} = t_2 - \frac{x_2}{a}$ after rearrangement, $a = \frac{x_2 - x_1}{t_2 - t_1}$

With the derivation presented, we confirmed our previous knowledge that the variable "a" in the argument is the ratio of the distance traveled by a selected point of the wavefront, x_2-x_1, and the elapsed time t_2-t_1, the speed of sound. The argument, in its specific mathematical language, expresses the propagating character of the wave. With a similar reasoning, the other component of the solution function, "g", describes the waves traveling in the minus x direction.

- In summary, the general solution of the wave equation describes free-propagating plane waves along the x-axis (in the positive and negative directions). There are two important physical meanings that arbitrary mechanical disturbance will propagate and the magnitude of the propagation velocity is "a".

- In space, a plane wave propagating in any optional n direction can be described by the following wavefunction,

$$p'(x,t) = f\left(t - \frac{\underline{r}\,\underline{n}}{a}\right) + g\left(t + \frac{\underline{r}\,\underline{n}}{a}\right)$$

3.3. Harmonic waves:

Harmonic excitation produces a harmonic wave. A harmonic wave (also known as a monochromatic wave or pure tone) can be described by a sine or cosine function. Among the solutions of the wave equation, the prominent significance of harmonic waves can be explained by the fact that the sine and cosine functions describing the harmonic wave are the basic elements of harmonic (spectral) analysis. In addition the free vibrations of the finite size flexible medium (e.g. air column in tube, cord tensed between two point, bell) are harmonic vibrations, or their composition, so the sound create by this vibrations will be a harmonic wave, or their composition.

In the argument of the sine and cosine function, suitable for describing harmonic motions, is an angular units (radians), so the time-based argument in the general solution must be changed to an angular units, which introduces the concept of the phase angle to characterise the state of the wave,

$$\omega\left(t-\frac{x}{a}\right)=\omega t-\frac{\omega}{a}x=\omega t-kx$$

The solution function f and g can be extended arbitrarily, so multiplying the argument by the value of the angular velocity (ω) is allowed (k is the wavenumber on the right side of the expression). Similar to harmonic oscillations, the value of the sound field variable at a given time and place for harmonic waves is equal to the projection value of an amplitude vector rotating at an angular velocity ω on the real axis x. In the case of vibrations, the angular position (phase angle) of the rotating amplitude vector depends only on time, but in the case of waves, the phase angle of the rotating vector is determined by the time and location coordinates together. From the last term in the expression of the transformed argument, it is clear that at any point at a distance x from the starting point, the phase angle must be reduced (retarded) by kx to determine the sound field variable. Taking these into account, the wave function of a harmonic sound wave with an amplitude $\hat{p}$, and an angular velocity ω traveling in the positive x direction for the sound pressure variable,

$$p'(x,t)=\hat{p}\cdot cos(\omega t-kx+\varphi_0)$$

Where

Notation	Meas. unit	Name	Physical meaning
p'(x,t)	[Pa]	sound pressure	pressure difference from the equilibrium value in sound field
$\hat{p}$	[Pa]	sound pressure amplitude	the magnitude of the largest deviation from the equilibrium value
$\omega t-kx+\varphi_0$	[rad]	phase angle	position of rotating amplitude vector
ω=2π/T	[rad/s]	angular velocity, angular frequency	the phase angle travelled by the wave per unit time
T	[s]	time of period	at x= const. place the elapsed time between two adjacent identical phase states of the wave (e.g. the time between two adjacent positive maximum)
f=1/T	[Hz]	frequency	number of periods per unit time
k=2π/λ	[rad/m]	wave number	the phase angle travelled by the wave per unit length
λ	[m]	wave length	at t= const. time the distance between two adjacent identical phase states of the wave (e.g. distance between two adjacent positive maximum)
φ_0	[rad]	initial phase angle	at t= 0sec time and at x= 0m position, the angle adjusting any phase of the wave

In harmonic waves, a surface containing fluid particles in the same perturbation state is called, as phase surface, due to the same phase angle of the rotating vector. Let the initial phase be φ_0= 0 rad and extract the angular velocity from the argument of the harmonic wave function,

$$\omega t-kx=\omega\left(t-\frac{x}{\omega/k}\right)=\omega\left(t-\frac{x}{a_f}\right)\ ,\ \ a_f=\frac{\omega}{k}=\frac{2\pi/T}{2\pi/\lambda}=\frac{\lambda}{T}$$

Where "a_f" is the phase velocity of the harmonic wave, the propagation velocity of the perturbation state for a given phase.

The complex exponential representation of harmonic waves: In derivations related to harmonic waves, exponential description is in many cases more advantageous than trigonometric (e.g. in the case of derivation, integration). To write the complex sound pressure, we supplement the wave function with an imaginary term, rewriting the trigonometric shape exponentially, separating the place- and time dependent and non dependent terms, the complex sound pressure (**p'**(x, t))

$$\boldsymbol{p}'(x,t) = \hat{p} \cdot cos(\omega t - kx + \varphi_0) + i \cdot \hat{p} \cdot sin(\omega t - kx + \varphi_0) = \hat{p} \cdot e^{i(\omega t - kx + \varphi_0)} =$$

$= \hat{p} \cdot e^{i\varphi_0} \cdot e^{i(\omega t - kx)} = \hat{\boldsymbol{p}} \cdot e^{i(\omega t - kx)}$ where the complex sound pressure amplitude, $\hat{\boldsymbol{p}} = \hat{p} \cdot e^{i\varphi_0}$

The complex sound pressure amplitude includes the initial phase in addition to the largest pressure deviation. However, only the real part of the complex sound pressure has real physical meaning. Thus, at the end of the derivations, after taking the formal mathematical advantages, the real part of the complex quantity must be taken in order to determine the true sound field characteristic.

$$p'(x,t) = Re(\boldsymbol{p}'(x,t))$$

3.4. Test questions and solved problems

T.Q.1. Based on the governing equation of the fluid mechanics derive in the homogeneous acoustic wave equation for the sound pressure variable! List the simplifications, explain the neglected terms in details! Put down the general plane wave solution of the equation! What are the importance and application of the equation and its solution!

S.P.1. In a long tube, a membrane performs a piston-like harmonic oscillation at a frequency of f= 225 Hz. The maximum speed of the membrane (v_{max}) is 0.006 m/s. At the initial moment, the membrane is in the middle position and moves at maximum speed towards the right part of the tube. Determine the particle velocity inside the air-filled tube, 226 m to the right of the membrane and 145 sec later than the initial time. The air temperature (t) is 250 °C, the adiabatic constant (κ) is 1.4.

$a = \sqrt{\kappa R T_0} = \sqrt{1.4 \cdot 287 \cdot (273 + 250)} \approx 458.4\ m/s$

$\omega = 2\pi f = 2 \cdot \pi \cdot 225 \approx 1413.7\ \ rad/sec$

$k = \omega / a \approx 1413.7/458.4 \approx 3.1\ \ rad/m$

φ_0= 0 rad

$v'(x,t) = \hat{v} \cdot cos(\omega t - kx + \varphi_0) = 0.006 \cdot cos(1413.7 \cdot 145 - 3.1 \cdot 226 + 0) = 0.0046\ \ m/s$
